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THE MATHEMATICAL DISCOURSE OF UNDERGRADUATE MATHEMATICS MAJORS: THE RELATION TO LEARNING PROOF AND ESTABLISHING A LEARNING COMMUNITY

A Dissertation Submitted to the School of Graduate Studies and Research in Partial Fulfillment of the Requirements for the Degree Doctor of Education

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August 2009

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Title: The Mathematical Discourse of Undergraduate Mathematics Majors: The Relation to Learning Proof and Establishing a Learning Community

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This study addressed the inherent and age-old quandary of learning mathematical proof. The aim of the study was to explore the nature of the learning of mathematical proof by undergraduate mathematics majors through the lens of discourse. Additionally, the study investigated mathematics majors' sense of a learning community in relation to their participation in a seminar on learning mathematical proof utilizing small-group discourse. A communicational approach to cognition—or commognition—provided the theoretical and research perspective for the study.

The setting of the study was a zero-credit seminar focusing on mathematical proof for freshman and sophomore mathematics majors. The primarily qualitative study had nine participants. A multiple methods strategy of data collection was employed. First, audio recordings of small-group discourse on mathematical proof were collected along with participants' related work. Participants additionally completed the *Classroom Community Scale* survey. Finally, interviews were conducted. Focal and preoccupational analyses were performed on the audio data to determine the object-level and meta-level features of the mathematical discourse/learning. Descriptive statistics and typological analysis were used respectively to summarize the survey and interview data.

A synthesis of these analyses revealed the complexity of learning mathematical proof; that is, of becoming a more expert participant in the discourse of mathematical

proof. Small-group discourse appears to be a comfortable way for novice interlocutors to *approach* a more expert discourse on proof. Moreover, there may exist in discourse between novice interlocutors natural opportunities, called *discursive entry points*, in which experts could intervene to steer the discourse towards increasing sophistication. Additionally, the study revealed several complex and interrelated factors related to learners' thinking (communication) of mathematical proof. The factors include: discursive contributions/role of interlocutors, discursive foci of interlocutors, difficulty/familiarity of mathematical content, negotiating effective communication, commognitive conflict, and power. Finally, interlocutors had a sense of *community* in the seminar on mathematical proof utilizing small-group discourse. The discourse may also have contributed to the *connectedness* that participants felt with their fellow math majors both inside and beyond the seminar walls. Moreover, the participants viewed being able to *communicate* about mathematical proof as the conduit to a universal math community.

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CHAPTER 1

IDENTIFICATION OF THE RESEARCH PROBLEM

Problem Statement

Student difficulty with mathematical proof has been widely documented. Procedural, conceptual, and communicative issues reoccur in the literature as particularly problematic areas. Baker and Campbell (2004) observed, for example, that their students struggled with understanding the process of proof construction, the precision involved in writing in mathematics, and the application of rules of logic in proof construction. Moore (1994) found that conceptual understanding, mathematical language and notation, and getting started on proof were the three major sources of mathematics and mathematics education students' difficulties. Weber (2001) hypothesized types of strategic knowledge that undergraduates lacked in abstract algebra proofs, including knowledge of the domain's proof techniques, knowledge of which theorems are important and when they will be useful and, knowledge of when and when not to use syntactic strategies.

Clearly, the difficulties faced by undergraduates in the learning of mathematical proof abound. Despite all we know about this challenge, the inductive and non-linear nature of discovering proofs make them hard to teach (Hale, 2003). As a result, undergraduate students often passively observe professors doing proofs throughout their first two years in lower division courses before making the startling and abrupt transition to needing to complete proofs on their own in upper division courses. The difficulty often evokes student frustration, as they understand neither the "why" nor the "how" of mathematical proof. Lutzer (2005) acknowledges the "why" question as legitimate and offers his own interpretation of the answer. Proof, he says, is important for determining circumstances in which an idea or interpretation leads to a correct conclusion or for determining whether a pattern truly exists or is simply an artifact of a specific situation. The "how" question is a bit more elusive. Understanding how *mathematicians* think about proof and a background on the philosophy of mathematics may narrow our focus and provide some clues about learning and teaching mathematical proof.

To gain insight into how mathematicians create mathematics, Sriraman (2004) interviewed five mathematicians who worked at large Ph.D. granting universities. With publications and research experience, each of the mathematicians had proven themselves in their field. Analytic induction of the interview data indicated that the four-stage Gestalt model of preparation-incubation-illumination-verification provides a framework for understanding mathematical creativity. Additionally, for these mathematicians, engaging in social interaction, imagery, heuristics, and intuition very often preceded proof construction. Sriraman notes that the mathematicians' approach to proof is very different from the logical approach presented by most textbooks. In a study on proof validation, Weber (2008) found that mathematicians use a variety of strategies. These include formal reasoning, the construction of rigorous proofs, informal deductive reasoning, and example based reasoning. Additionally, the mathematicians' conceptual knowledge, the mathematical domain of the proof, and the status of the proof's author were important factors in validation. Certainly, an understanding of the processes involved in mathematical productivity are important for discussing the epistemology of mathematics and the closely intertwined teaching and learning of mathematics.

The search for a flawless philosophy of mathematics is storied and ever evolving. It begins with Plato, who held the viewpoint that purely intelligible objects are not of this world and consequently unattainable by worldly experience (Honderich, 1995). Thus, in Platonism, the process of learning mathematics is akin to searching for an objective truth. In the nineteenth century, mathematical developments spawned three new philosophical schools of thought. In *logicism*, the way to analytic truth is through the laws of logic and definitions. Logic then, forms the foundation of all mathematical truth. *Formalism*, takes mathematics as a game. The symbols of mathematics are meaningless. What counts is how the mathematician maneuvers them using the rules of the game. Among other inconsistencies, mathematical in nature and beyond the scope of this discussion, Platonism, logicism, and formalism are all inadequate in addressing the practice of mathematicians. While there may exist no ideal philosophy, the nineteenth century philosophy of *intuitionism* is more consistent with the human elements of mathematics. For intuitionists, mathematics is not independent of human thinking, but rather found in the mental constructions of human thinking. Intuitionism then, is a brand of mathematical constructivism.

Throughout the twentieth and now the twenty-first century, the human and social aspects of creating mathematics have received much attention. The works of Polya (1954) and Lakatos (1976), for example, challenge the notions of formalism. Polya makes the case that, in mathematics, plausible reasoning—the provisional reasoning of conjectures—is no less important than demonstrative reasoning—the final and unquestionable reasoning of formal logic. He argues:

> Mathematics is regarded as a demonstrative science. Yet this is only one of its aspects. Finished mathematics presented in a finished form appears as purely demonstrative, consisting of proofs only. Yet mathematics in the making resembles any other human knowledge in the making. You have to guess a mathematical theorem before you prove it; you have to guess the idea of the proof before you carry through the details. You have to combine observations and follow analogies; you have to try and try again. The result of the mathematician's creative work is demonstrative reasoning, a proof; but the proof is discovered by plausible reasoning, by guessing. If the learning of mathematics reflects to any degree the invention of mathematics, it must have a place for guessing, for plausible inference. (p. vi)

Lakatos argued that formalism is a distilled, final-product view of mathematics that egregiously omits its own history—the history that Isaac Newton referred to when he said he had seen further because he stood on the shoulders of giants (Steen, 1990). For Lakatos, "mathematics does not grow through a monotonous increase of the number of indubitably established theorems but through the incessant improvement of guesses by speculation and criticism, by the logic of proofs and refutations" (p. 5). Most recently, the social constructivist philosophy of mathematics, has received increasing attention (Ernest, 1998; Hersh, 1997). Here, mathematical objects are social entities created through an ongoing conversation, spanning culture and time. The re-conception of mathematical knowledge throughout the last century just described has definite implications for the teaching and learning of mathematics.

Lecture-based courses have traditionally dominated the mathematical sciences. Typically, students have encountered mathematics classrooms that are teacher-centered (Berger, 1996). These traditional classrooms often follow an instructional pattern wherein the instructor *initiates* a question, awaits a brief *reply*, and *evaluates* the reply (from this point forward called IRE) (Mehan, 1979). College-level instructors may favor such instruction because of the failure of students to ask or answer questions; a feeling of responsibility to control the outcome of the discussion or to progress somewhere with an idea or text; an awkwardness with the potential silence; and the unpredictable nature of discussion. Despite these challenging realities of teaching, Neal (2008) argues:

> the real enemy of intellectual rigor and of students' development of academic authority is . . .linguistic exclusion, which is often created by classroom discourse that controls and limits students' opportunities to talk with us and with each other, to question, and to think out loud. (p. 280)

Classrooms in which discourse is one-way, from teacher to student, can leave students with impressions of mathematics that are inconsistent with the practice of mathematicians. They may, for example, develop the notion that mathematics is a body of knowledge to be transmitted or conveyed rather than created. Furthermore, they may view mathematics as a meaningless set of rules to be carried out.

Not only is teacher-dominated discourse in the mathematics classroom exclusionary, it also conveys a questionable absolutism of the discipline. Ernest (1998) argues that a mathematical philosophy accounts for the "social construction of the individual mathematician and her/his creativity, if it is to account for mathematical knowledge naturalistically" (p. xiii). Indeed, as indicated in the Sriraman study (2004), mathematicians engage creatively in their work using social interaction, preparation, heuristics, imagery, incubation, illumination, verification, intuition, and proof. Unfortunately, mathematics classrooms do not always reflect this process.

Addressing college mathematics curricula for both mathematics and non-mathematics majors, the 2004 Curriculum Guide of The Mathematical Association of America (MAA) recommends that:

> Every course should incorporate activities that will help all students progress in developing analytical, critical reasoning, problem-solving, and communication skills and acquiring mathematical habits of mind. More specifically, these activities should be designed to advance and measure students' progress in learning to
>
> - State problems carefully, modify problems when necessary to make the tractable, articulate assumptions, appreciate the value of precise definition, reason logically to conclusions, and interpret results intelligently;
> - Approach problem solving with a willingness to try multiple approaches, persist in the face of difficulties, assess the correctness of solutions, explore examples, pose questions, and devise and test conjectures;
> - Read mathematics with understanding and communicate mathematical ideas with clarity and coherence through writing and speaking (Barker et al., p. 6)

The recommendations speak to a range of human activities in which those in the discipline engage. It would seem that traditional models of instruction are ill-suited for giving *students* the opportunities to engage in these activities and consequently for accomplishing the aforementioned curricular ideals. A classroom in which students are active participants in mathematical discussion—posing questions and providing explanations—is more congruous with the MAA's vision for student learning.

In fact, classroom discourse can play a vital role in undergraduate education. For coursework within a student's major, classroom discourse serves to socialize the student into a professional identity (Dannels, 2000; Northedge, 2003). What is more, research (e.g., Johnson, Johnson & Smith, 1991; Smith 1977; Terenzini, Theophilides & Lorang, 1984; Tsui, 2002) indicates that the amount of cognitive level of student participation and the amount of interaction among students within a course are consistently and positively related to gains in critical thinking, academic skills, and student motivation. Springer, Stanne, and Donovan (1999) report similar findings—greater academic achievement, more favorable attitudes toward learning, and increased persistence—in a meta-analysis on the effects of small-group learning, specific to courses and programs in Science, Technology, Engineering, and Mathematics (STEM). Yet, Nunn (1996) found that while both college students and teachers similarly value and desire classroom interaction, there is little time devoted to it in the college classroom. Furthermore, the college teachers in Nunn's study indicated feeling less skilled at leading discussions than teaching using a traditional format.

Instructors of mathematics at the undergraduate level who have renounced a lecture format in favor of more interactive methods have reported on its advantages.

Berry and Sharp (1999), for example, discuss the results of using a transformation model, as opposed to transmission model, in teaching mathematics in higher education. A key finding of their study is students' increasing awareness of learning mathematics by doing instead of watching. Additionally, the researchers noticed improvement in written solutions on formal examinations during the three years in which they applied the active learning module. King (2001) reports favorably on his experiences using a lecture-free seminar format at all levels of the undergraduate mathematics curriculum. His lecture-free method consists of pre-seminar readings, pre-seminar exercises, pre-seminar reaction pieces, seminar class discussion, and post-seminar problem sets. Anecdotally, King notices that the seminar style courses allow students to improve their ability to learn mathematics independently and to take pride in their mathematics achievements. Furthermore, King suggests that the seminar format may assist students who have a weak background in mathematics by improving their mathematics study skills and motivation.

Combined, these studies suggest that students develop a deep understanding of mathematics, as opposed to a surface understanding, when given the opportunity to engage in the conversation of mathematics. What, if any, are the features of classroom discourse that lead to student learning of mathematical proof? What implications might knowing this have on the teaching and learning of mathematical proof for undergraduate students?

To begin to answer these questions, it is necessary to have an agreed upon understanding of what it means to learn something. Sfard's (2002, 2008) commognitive approach to thinking and learning serves as the theoretical framework for this study. In this approach, thinking is a variant of communicating. Accordingly, the rules that govern effective communication and lucid thinking are interchangeable. To learn mathematics is to participate in its discourse—employing the tools and the rules that distinguish the communication as mathematical. Chapter 2 elaborates on commognition.

Purpose of Study

The primary motivation for this study is the persistent difficulty that students of mathematics have in learning mathematical proof. It is plausible that the end-product view of proof that they often encounter may contribute to this difficulty. The monologic form of proofs in texts, for example, quiets the listener (reader/learner) by presenting an argument that anticipates all possible objections (Ernest, 1994). Similarly, traditional lecture-based mathematics classrooms tend to convey perfection and absolutism. Mathematical proof, however, traces back to Classical Greece where disputation and dialectical reasoning were common practices (Ernest). Not surprisingly, then, present day research highlights the role of social interaction in mathematical creativity (Sriraman, 2004). Moreover, engaging students in mathematical communication is an essential component of current reform in mathematics education (Barker, 2004; NCTM, 2000). The purpose of the study stems from this backdrop. Its aim is to explore the learning of mathematical proof by undergraduate mathematics majors through the lens of mathematical discourse.

Recent philosophies of mathematics that account for the social construction of mathematical knowledge (Ernest, 1998; Hersh, 1997) inform the purpose of the study as well. The study considers the overall community context in which the mathematical discourse occurs. Sriraman (2004) contends, "The types of questions asked [by mathematicians] are determined to a large extent by the culture in which the mathematician lives and works" (p. 21). Thus, an additional intention of the study is to explore the nature of a learning community as it relates to the learning of mathematical proof by undergraduate mathematics majors engaging in small-group discourse.

Research Questions

The study addresses two research questions. They are:

1. What is the nature of the relationship of undergraduate mathematics majors' discourse of mathematical proof to their learning of mathematical proof?
2. What is the nature of undergraduate mathematics majors' sense of community in a seminar utilizing a small-group discourse format for the learning of mathematical proof?

Definitions of Terms

Commognition—a term that encompasses *thinking* (individual cognition) and (interpersonal) *communicating*; as a combination of the words communication and cognition. It stresses the fact that these two processes are different (intrapersonal and interpersonal) manifestations of the same phenomenon (Sfard, 2008, p. 296).

Discourse— A special type of communication made distinct by its repertoire of admissible actions and the way these actions are paired with re-actions; every discourse defines its own *community of discourse;* discourses in language are distinguishable by their vocabularies, visual mediators, routines, and endorsed narratives (Sfard, 2008, p. 297).

Learning Community—This study adopts a "purposeful" notion of community in higher education. In particular, it emphasizes the importance of building community around a common academic purpose. It additionally recognizes the unique potential that an academic department has to become a "creative intellectual social unit on the campus through special seminars, lectures, and social events for students and faculty" (Carnegie Foundation, 1990, p. 13). Moreover, within the commognitive framework, a *discourse community* encompasses all of those capable of participating in a given discourse (Sfard, 2008, p. 297).

Learning Mathematics—An initiation to mathematical discourse—the special form of communication that is mathematical. Skillful participation in mathematical discourse requires a command of its object-level rules and meta-discursive rules (discussed in detail in Chapter 2) (Sfard, 2002, p. 28). Thus, the mark of learning is the changing of discourse in a lasting way (p. 299). In summary, "Thinking is conceptualized as a special case of the activity of communication and learning mathematics means becoming fluent in a discourse that would be recognized as mathematical by expert interlocutors" (Sfard, Forman & Kieran , 2002, p. 5).

Mathematical proof—The logical organization of the evidence that a theorem is true (consisting of rules of logic, previous steps in the proof, previous theorems proved, axioms, and definitions) (Hale, 2003). Within the commognitive framework, proof is understood as a sequence of endorsed narratives, each of which is deductively inferred from previous ones, and the last of which is the narrative that is being endorsed (Sfard, 2008, p. 232).

Significance of Study

This study adopts Sfard's (2002, 2008) commognitive approach to thinking and learning, which is explained in detail in Chapter 2. Briefly, the approach is rooted in sociocultural psychology, which views learning as becoming a *participant* in distinct activities. Its focus is on the interactional and contextual aspects of learning, as opposed to the cross-cultural invariants of interest to cognitivists. Sfard argues that the cognitivist acquisition model of learning has proven inadequate in explaining tough issues in mathematics education, not the least of which is the persistent failure of some students to learn mathematics. On the other hand, the participationist framework provides a hopeful outlook on learning. Instead of focusing on intangibles, such as students' mental schemas, we can focus on something that is alterable—the social context.

The teaching and learning of mathematical proof continues to confound teachers and students of mathematics. To this point, however, research on learning mathematical proof has focused on a daunting list of what students lack—procedural knowledge, strategic knowledge, conceptual understanding, a command of mathematical knowledge, a desire to understand why mathematical statements are true, and the list goes on. The primary significance of this study then, is its novel approach to examining the learning of mathematical proof through the lens of mathematical discourse. A careful analysis of the contextual data has the potential to inform ideas about fostering effective mathematical communication and, in tandem, mathematical learning in relation to mathematical proof.

The limitations of traditional methods of instruction have been widely acknowledged. Theory rather than empirical evidence, however, guides the methods emphasizing interactive approaches to learning recommended in the place of traditional methods. This is not to say that interactive approaches are inherently bad or worse than traditional methods. What it does imply is that we cannot take learning for granted when using interactive methods. What elements of interaction make learning most effective? What elements of interaction inhibit learning? There exists a glaring lack of research examining mathematics classroom discourse at the collegiate level. By closely examining the learning of mathematical proof of undergraduate students through interactive discourse, this study should inform the successful enactment of reform-based teaching recommendations at the secondary and post-secondary levels.

Limitations of the Study

An unavoidable limitation of this study is time. The study setting—a freshman/sophomore mathematics seminar during the fall 2008 semester—provided an opportunity to analyze small-group discourse and the closely related learning of mathematical proof. The study does not include an analysis of the role that the seminar will play in students' inevitable increasing sophistication in proof in successive coursework nor in their continued participation in a mathematical learning community.

A delimiting factor of the study is that the seminar under investigation is a zero-credit seminar. The seminar carries a course number and student participation is strongly encouraged within the mathematics department. However, faculty cannot mandate participation. Thus, the subjects of this study may possess different motivations for participation in learning about mathematical proof than students enrolled in a credited course.

Because the study is limited to a specific group—a freshman and sophomore zero-credit seminar on proof for mathematics/mathematics education undergraduate students—its conclusions can only be applied to this local context. Thus, the research design limits the possibility of developing a general theory applicable across research contexts. However, what is lost in generalizability is made up for in the rich description of data from a “high-resolution” approach to analysis. The reader is left to judge how the findings from this study might inform other contexts.

Summary

This chapter has highlighted the vexing difficulties that have inhibited the teaching and learning of mathematical proof. The discussion of the practice of mathematicians and reigning philosophies of mathematics provides fodder for thinking about engaging students in the discourse of the discipline. Furthermore, the commognitive approach to cognition, offers a promising and compatible way for exploring the relationship between mathematical discourse and the learning of mathematical proof. In the next chapter, the commognitive approach is placed in a much broader discussion of competing constructivisms. Additionally, pertinent literature related to mathematical discourse, mathematical proof and, mathematical learning communities is reviewed.

CHAPTER 2

REVIEW OF THE LITERATURE

This chapter begins with an overview of constructivism and its two major perspectives—cognitive constructivism and social constructivism. Next, recent efforts in mathematics education to coordinate the two perspectives are explored. The chapter then includes a detailed discussion of *commognition*, or thinking as communicating, which serves as the theoretical framework for this study. The second half of the chapter continues with a general discussion of mathematical discourse followed by a discussion of mathematical discourse within a commognitive framework. A summary of the research on learning mathematical proof and a discussion of learning communities and their role in higher education precede the conclusion, which synthesizes the key ideas of the chapter.

Constructivism

While behaviorism was the dominant learning theory for most of the 20th century, constructivism gained prominence during its last two decades (Ornstien & Hunkins, 2004). An increased interest in cognitive science and human information processing during the latter portion of the century contributed to this shift (Ornstien & Hunkins; Palinscar, 1998). Behaviorism focuses on the role of an external force in eliciting an observable response from the learner. Because of its attempts to do away with "mentalistic notions" such as *meaning, representation*, and *thought*, behaviorism has come under much criticism for blurring the distinction between training and education (von Glaserfeld, 1987). Constructivism, on the other hand, places primacy on the learner as *active* in the process of thinking, learning, and coming to know. In other words, "human knowledge—whether it be the bodies of public knowledge known as the various disciplines, or the cognitive structures of individual knowers or learners—is *constructed*" (Phillips, 1995, p. 5).

Among constructivists, however, there is disagreement on *how* the individual learner constructs knowledge. The next two sections describe the two major schools of thought—cognitive constructivism and social constructivism—that have developed around this central question. They additionally explore the influence of each school of thought within mathematics. Before proceeding, it is worth noting that the literature uses a variety of terms to describe the two approaches. For example, some authors refer to cognitive constructivism as the constructivist perspective. Similarly, some authors refer to social constructivism as the sociocultural perspective or interactionism. For clarity's sake, the current discussion uses the umbrella terms of cognitive and social constructivism to distill the variants of a very broad discussion into its most essential elements.

Cognitive Constructivism

Theorists operating within the cognitive constructivism paradigm attribute learning to the sensory-motor and conceptual activity of the individual. The cognitive constructivists draw heavily from Jean Piaget's genetic epistemology, which attempted to relate the validity of knowledge with models of knowledge construction (1970). The epistemology encompasses Piaget's four stages of development (sensorimotor, pre-operational, concrete operational, and formal) as well as an explanation of the process of passing from one developmental stage to the next (assimilation, accommodation, equilibrium).

Piaget (1971) also recognized the social dimension of development. He wrote: "Human intelligence develops in the individual in terms of social interactions too often disregarded" (p. 224-225). Drawing from his theoretical approach, Neo-Piagetian theorists have documented that interpersonal conflicts between learners create internal individual conflicts (Doise & Mugny, 1979; Doise, Mugny & Perret-Clermont; 1975). It is through the internal resolution of these conflicts that learning occurs. In other words, when outside experiences contradict a learner's existing understanding disequilibrium results forcing the learner to develop new cognitive structures. Piaget (1985) suggested that cognitive development was most probable in social interactions between peers because of the mutual control over the interaction. In summary then, for cognitive constructivists, knowledge takes the form of cognitive structures organized in memory. Moreover, while learning is influenced by the social context, it is ultimately a process of active individual construction.

Von Glaserfeld (1987) offered the mathematics education community a cognitive constructivist model of knowing. Serving as a pedestal for the model is the idea that knowledge is a derivation of human experience. Furthermore, the litmus test for the value of this experientially derived knowledge or conceptual structure, is what von Glaserfeld calls its *viability*. To be viable, the knowledge must "fit", as opposed to conflict, with prior experience. This model is in contrast with an "iconic" one in which knowledge or cognitive structures are perfect "matches" with the structures they are to represent from an independent and objective reality. For von Glaserfeld, the elusive nature of reality makes this conception problematic. Treating communication as a specific type of experience, he challenges the notion that words are symbolic containers for writers and speakers to convey an objective meaning. In keeping with his model of knowing, von Glaserfeld outlines a composite of experiences from which an individual abstracts the concept associated with a word. To learn a word, we must first isolate its recurrent sound pattern from the entirety of sensory signals. Next, we must distinguish something recurrent in the experiential field that occurs in combination with the sound pattern. And finally, we must be able to represent the concept in our minds when we hear the word, without the aid of the experiential field.

Questions of communicability are inherent to this model. If individuals construct concepts from their own unique experiences, how do we know that speakers and listeners share the same representation? Von Glaserfeld (1987) suggests that it is a mistaken assumption that the representations between communicators must be the *same.* Understanding is achievable as long as the representations do not conflict with the communicators' situational context or their expectations.

> If our interpretation of experience allows us to achieve our purpose, we are quite satisfied that we "know"; and if our interpretation of a communication is not countermanded by anything the communicator says or does, we are quite satisfied that we have "understood." (p. 10)

For von Glaserfeld, understanding is a matter of building conceptual structures that work within experiential constraints. The structures of mathematical concepts, however, tend to be obscure—with mathematical definitions relying on symbols. The difficulty is that these symbols provide few clues as to what is necessary for building related conceptual structures. However, under the model, we cannot simply transmit or deposit mathematical knowledge in the minds of learners. Neither can we speak of mathematical concepts as if they existed independently in an objective reality.

To move students to understanding of mathematical concepts, von Glaserfeld (1987) says that teachers must foster their reflective awareness (the ability of the mind to observe its own operations). To accomplish this, von Glaserfeld favors the teaching experiment where the experimenter generates a viable model of the child's present concepts and operations, hypothesizes pathways to guide the child's conceptualizations toward adult competence, and then applies indirect guidance to modify the child's present concepts toward the adult concepts. Indirect guidance consists of bringing about some conflict of experience or expectation for the student. Guidance takes the form of "either questions or of changes in the experiential field that leads the child into situations where her present way of operating runs into obstacles and contradiction" (p. 14). Presumably, conflict results in an internal reorganization in which students' cognitive structures become once again viable.

Social Constructivism

Whereas cognitive constructivists focus on the construction of knowledge in the "head" of an individual, social constructivists emphasize the role of social and cultural contexts in the co-construction of knowledge. Social constructivist thinking builds on the work of Lev Vygotsky's social-historical theory of cognitive development. In describing internalization—the process by which an individual absorbs knowledge from his or her external setting—Vygotsky wrote:

> Every function in the child's cultural development appears twice; first, on the social level, and later, on the individual level; first, *between* people (*interpsychological*), and then *inside* the child (*intrapsychological*). This applies equally to voluntary attention, to logical memory, and to the formation of concepts. All the higher functions originate as actual relations between human individuals. (Vygotsky, 1978, p. 57)

Thus, studying cognitive development from a Vygotskian perspective entails the examination of how shared engagement in an activity is transformed into internalized processes (Palincsar, 1998).

Vygotsky was particularly interested in the psychological tools, what he called cultural *signs*, which people use to aid their thinking (Crain, 2005). Among the most important sign systems are speech, writing, and numbering systems. While Vygotsky accepted that intrinsic maturational cognitive development does occur, especially from birth to the age of 2 years, he believed that the culture's sign systems exhibit the principal influence on cognitive development. Palincsar (1998) identified several research findings that are motivating a current interest in social constructivism. All of them highlight the connection between signs (e.g., language) and learning. The following strategies have been found helpful for learning.

- *Expert reasoning and problem solving*. In classrooms, this translates to the use of public modeling via think-aloud and reciprocal teaching.
- *A collective memory.* In classrooms, this translates to groups attaining more success than individuals working alone.

- *Language production.* Explanation of one's thinking leads to deeper cognitive processing.

For social constructivists, knowledge is socially derived and learning is situated in the external processes of interaction, negotiation, and collaboration.

Modern thinking has tended to move beyond a Platonic view of mathematics—one in which objects purely intelligible are not of this world and consequently unattainable by worldly experience (Honderich, 1995). One alternative philosophy gaining in popularity is social constructivism (Ernest, 1998; Hersh, 1997). In general, social constructivism recognizes mathematical activity as a human activity; mathematicians construct mathematics through conversation. Thus, social constructivism treats mathematical objects as social entities rather than physical or mental entities. Hersh argues that discovering and defining a mathematical object is a social process. However, once the mathematical community defines an object, it achieves an immutable and non-social status. In other words, mathematicians no longer have power to change it. In contrast, for Ernest, mathematical definitions reside in the agreement of the mathematical community. As a philosophy of mathematics, Platonism failed, in part, because of its inability to account for the experiences of mathematicians in their daily work of mathematical research. While social constructivism may also have its flaws as a philosophy of mathematics (see for example Gold, 1999), it necessarily highlights the human element that was missing from Platonism.

Bridging the Gap Between Cognitive and Social Constructivisms

Both cognitive and social constructivisms are influential in present day mathematics education research. Mathematical learning is treated as a process of active individual construction in some cases and principally as a process of acculturation in others. There exist within the mathematics education research community, however, efforts to consider cognitive and social constructivisms as complementary (Bauersfeld, 1988; Cobb and Bauersfeld, 1995; Sfard, 2002; Sfard, Forman, & Kieran, 2002). The movement to harmonize the approaches is not unique to mathematics education. Martin and Sugarman (1997), for example, offer a theory for psychotherapeutic change that combines social constructionist and cognitive constructivist thinking. And more recently, Felix (2005) asserted the complementary nature of the two perspectives for language learning in online learning environments.

Cobb and Bauersfeld (1995) identify both the values and limitations of each perspective in relation to an improved understanding of mathematical learning. For example, they note the important contribution of social constructivism in the areas of cultural and ethnic diversity during the current reform era of mathematics education. However, they also take the position that theories developed in the Vygotskian tradition relate only to the *possibility* of learning. In other words, Cobb and Bauersfeld view the examination of the external processes of learning without the examination of individual students' cognitive activity as inadequate. At the same time, they concur with Solomon's (1989) critique of a purely cognitive approach. A learner could resolve a mathematical cognitive conflict in any number of ways. Only some of which, however, are compatible with the agreed upon meaning of the mathematical community. Thus, the construction of knowledge and the socio- cultural context are inextricably intertwined.

In their edited volume, *Emergence of Mathematical Meaning: Interactions in Classroom Cultures,* Cobb and Bauersfeld (1995) seek to transcend the dualism that cognitive and social constructivism present by "coordinating sociological analyses of the microculture established by the classroom community with cognitive analyses of individual students' constructive activities" (p. 7). Their coordination admittedly falls short of a seamless theoretical framework. Rather they offer a notion of *reflexivity* in which "neither an individual student's mathematical activity nor the classroom microculutre can be adequately accounted for without considering the other" (p. 7). Examples of reflexive relationships in a mathematics classroom might include the students' mathematical activity and the social relationships they establish, the relation between the quality of a student's explanation and the social situation in which it is developed, the relation between mathematical themes and individual contributions, and engaging in learning and argumentation.

In practice, the notion of reflexivity is comparable to the "zoom function" on any number of electronic devices (e.g., graphing calculator, digital camera or computer screen). As the researcher "zooms in" on an individual student's cognitive activity, the social context will fade into the background. However, as the researcher "zooms out" to capture the social context, the characteristics of individual activity will no doubt be less discernible. What is important in the notion of reflexivity, is that the researcher never loses sight of the mutuality that exists between the sociocultural context and the individual's constructive activities.

Noddings (1990) asserts that constructivism is not only a powerful theoretical position, but also a powerful methodological perspective. Methodological constructivism is a natural consequence of the acceptance that all knowledge is constructed. It requires a complex examination of subjects' perceptions, purposes, premises, ways of working things out, and interactions with their physical and cultural environments. In other words, the methods of study must correspond with the premise that all knowledge is actively constructed. With a growing acceptance of the theoretical notion of reflexivity, what then, are the implications for research?

Cobb (1994) argues: “In place of attempts to subjugate research to a single, overarching theoretical scheme that is posited *a priori*, we might. . . reflect on and document our attempts to coordinate perspectives as we attempt to cope with our specific problems” (p. 19). Working from the premise of reflexivity places an onus on researchers to be forthright. They must first acknowledge and justify their choice in theoretical positions (cognitive versus social). Furthermore, it is necessary to explain *why* the other position is *not* being used. For example, operating from a social constructivist perspective requires both an explication of why it is not important to focus on the individual cognitive activity for the particular research purposes and an acknowledgment of instances in which a cognitive approach *would* be appropriate. Ideally, this effort to coordinate the approaches produces a more inclusive recognition of the complex realities of practice.

Commognition

Sfard, Forman, and Kieran (2002) also tackle the seeming incompatibility of the cognitive and social constructivist perspectives within mathematics education. They begin with an acknowledgment of the inherent methodological difficulties that researchers face when the object of their investigation is the human mind. In particular, they note that a relentless effort to adhere to the methodological criteria of the “exact sciences” has historically resulted in a perversion of purpose. Behaviorism, which failed to address mental non-observables, is a case in point. In another example, despite their expectations, researchers have been unable to demonstrate invariance in mathematical cognitive processes across cultures. Thus, cross-cultural studies in mathematical thinking have proven the acquisition metaphor of cognitivism inadequate. For Sfard et al., behaviorism and cognitivism fall short in their explanatory power of mathematical learning. In particular, they view them as inadequate for resolving problems grounded in practice—such as the persistent failure of some students in learning mathematics.

Sfard (2008) specifically identifies five "quandaries" of mathematical thinking that persist despite the long history of research on thinking. The five quandaries include numerical thinking, abstraction (and transfer), misconceptions, learning disability, and understanding. Sfard suggests that the perpetuation of the quandaries very well may lie in how researchers have discussed thinking. She identifies as the principal historical culprit the *metaphor of object* or the "tendency for picturing the perceptually inaccessible world of human thinking in the image of material reality" (p. 42). It is not unusual in research or in practice, for example, to hear a statement such as "two [of my] students constructed similar conceptions of function" (p. 43). But conceptions (and learning disabilities and abstractions) are non-observables. When studying conception (or learning disability or abstraction) what the researcher actually observes is *people in action.* She states:

> Indeed, in the discourses on humans and their doings, reification and alienation [aspects in the process of objectification] may lead to *illusory dilemmas*—dilemmas that result from unfortunate metaphorical entailments, to *phony dichotomies* that engender tautological statements disguised as causal explanation, and to *consequential omissions* resulting from the fact that the "low-resolution"

> objectified descriptions of human phenomena gloss over important inter-personal and intra-personal differences. In addition, objectified discourses on thinking tend to produce diagnoses and evaluations that function as self-fulfilling prophecies. These and some other weaknesses of our current thinking about thinking are sufficient incentive for trying to ground the discourse of research in a more operational, disobjectified infrastructure (p. 64)

Sfard first posits the possibility of a theoretical research discourse that, "although effective in describing and organizing what we see when observing people in action, makes no reference to objects such as concepts, learning disability, or abstraction" (p. 43). She then goes on to offer such a theory—one that she calls *commognition*.

Theory

Sfard (2008) first grounds her theory building in historical attempts at disobjectification. The 20th century philosopher Ludwig Wittgenstein is particularly influential in her thinking. She writes:

> For Wittgenstien, meaning was neither a thing in the world nor a private entity in one's mind: It was an aspect of human discursive activity and, as such was public and fully investigable. (p. 73)

Sfard juxtaposes Wittgenstien's deep regard of the complexity of human actions against his famous proclamation that, "What we cannot speak about we must pass over in silence" (Wittgenstien, 1997). Specifically, she describes the category of "communicables" that Wittgenstein considered fully investigable as vast and highly complex. In making a case for commognition, Sfard also draws on the participationist vision of humans and their development, which began to emerge in the late 1980s. Here, Lave and Wenger's (1991) understanding of learning as *legitimate peripheral participation* in socially organized activities is significant. Sfard describes this transition from learning-as-acquisition to learning-as-participation as follows:

> Rather than being an acquirer of goods, the learner was now to be viewed as a beginning practitioner trying to gain access to a well-defined, historically established form of human doing. The term *socially organized* was not supposed to imply that the activities in question must always be performed in collaboration with others. It only meant that processes of learning, as other human activities, are part and parcel of a patterned collective effort.

Taken together, the constructs of disobjectification and learning-as-participation open the way for understanding thinking as an individualized form of communication—the basic premise of commognition.

Sfard (2008) presents the term *commognition*—a purposeful combination of the words communication and cognition—as an initial step in building a disobjectified discourse on thinking. To do so, she notes that what distinguishes humans from other species is not only the ability to think in complex ways but also the highly developed ability to communicate. Without interpersonal communication, human needs ranging from the most primitive biological ones to advanced cultural ones would go unmet. Accordingly, Sfard posits that there is utility in defining thinking as an individualized version of (interpersonal) communication because the formulation "leads to a rich, coherent, and cogent set of narratives about the defined phenomena" (p. 82). What does Sfard intend by the word communication? Given her theoretical framework, it is not surprising that she rejects the traditional definition of communication as the exchange of information, messages, thoughts, feelings, or meaning between two individuals. Instead, she takes communication as a "patterned collective activity that involves a repertoire of permissible (communicational) actions of individual members and, for each such action, a repertoire of permissible re-actions of other individuals" (p. 93). In other words, an outside observer of communication cannot *see* information passed between two individuals, but he/she can observe across time and across situations certain regularities in the actions and re-actions of those communicating.

Communicational patterns are not governed in some pre-determined way by natural laws. Rather, communication is possible because a community "*got into a habit* of reacting to certain actions with certain types of reaction" (Sfard, 2008, p. 88). This leads to a number of important characteristics of communication. First, the habits formed by a community serve as *de facto* rules that tend to constrain the possibilities for communicable actions and re-actions in a given situation. They do not, however, determine a singular response—and thus the door is left open for the "constant accumulation of complexity of human action" to which we seek so many answers (p. 89). Second, there exists a distinction between practical actions and communicational actions.

> Whereas practical actions are direct actions *on* objects, communicational actions are *about* objects; that is, they may *lead to* an action on an object or to another communicative action about an object. In any case, *the object of a communicational act* is a thing to which the actor drives the re-actor's attention. (p. 89)

Third, participants in communication rely on communication mediators or perceptually accessible objects that assist the actor in performing the prompting action and the re-actor in being prompted. “Although any material object can be adapted to serve in this communicational role, communicational mediators are often objects produced specially for the sake of communication” (p. 90). Symbols are one such example of specially created communicational mediators.

Potential Efficacy of a Commognitive Approach to Research

In light of the historical shortcomings of behaviorism and cognitivism, the commognitive framework can be a source of optimism for researchers of mathematical thinking and learning. Within this framework, “thinking is conceptualized as a special case of the activity of communication and learning mathematics means becoming fluent in a discourse that would be recognized as mathematical by expert interlocutors” (Sfard, Forman, and Kiearn, 2002, p. 5). Whereas traditional frameworks characterize learning as intellectual acquisition and thus a change in the individual learner, change in communication is key for understanding learning within the discursive framework. Thinking is conceptualized as communication with oneself. “Indeed, our thinking is clearly a dialogical endeavor, where we inform ourselves, we argue, we ask questions, and we wait for our own response” (Sfard, 2002, p. 26).

Two points are important for understanding how the commognitive approach works. First, Sfard treats the basic mechanisms of communication with oneself and communication with others as essentially the same. Thus, in this approach to cognition, the demands of effective communication are one and the same for effective thinking. Second, instead of understanding speech as an expression of thought, the two are considered inextricable aspects of the same phenomena. By centering the discussion on communication, Sfard and colleagues (2002, 2008) claim to “sidestep” the dichotomy of the cognitive and social constructivist perspectives. Instead of working against one another, the cognitive and social approaches are just two different ways of examining the same phenomenon.

The commognitive framework allows us to define mathematical learning as an initiation into mathematical discourse. Thus, the rules of discourse become the tools for understanding learning. Sfard (2002) is careful to point out that this approach should complement and supplement, not supplant, acquisition theories in cognition. However, she makes the case that considering communication indistinguishable from thinking will allow us to capture the contextual part of the story of learning previously sifted out by acquisition-based theories. The hope is that what we capture might inform practice in lasting and effective ways. Commognition serves as the theoretical foundation for the study under discussion. Thus, the next section takes an in-depth look at mathematical discourse. It provides an overview of frameworks for discourse presented by other researchers before presenting Sfard's definition of mathematical discourse and its rules.

Mathematical Discourse

Sherin (2002) describes the process of mathematical discourse as "the way that the teacher and students participate in class discussion" (p. 206). Although this definition is straightforward and unambiguous, mathematical discourse is anything but simple. It is necessarily colored by infinite complexities, such as the participants' relationships to one another and the situational context in which it occurs. Several researchers have developed frameworks for examining the intricacies of discourse. This section begins with an overview of two such frameworks offered by Knuth and Peressini (2001) and Krussel, Edwards, and Springer (2004). A discussion of Sfard's rules for discourse (2002; 2008) follows. Moreover, we will dismiss the notion that knowledge is some object to be transmitted from teacher to student. Instead knowledge is constructed (whether cognitively or socially). This has enormous implications for classroom discourse and student learning. Thus, levels of a math-talk learning community are presented as the backdrop for exploring this shift in thinking and its classroom implications (Hufferd-Ackles, Fuson, and Sherin; 2004).

A Dualistic Function of Discourse

Drawing on work in semiotics, Knuth and Peressini (2001) offer a dualistic theoretical framework for examining mathematical discourse in the classroom. The conception underlying the framework is that a listener must transform a speaker's words into a personal understanding. Thus, the quantity and quality of opportunities for interanimation influence opportunities for understanding. In *univocal discourse* interanimation is limited. Its primary functions are to convey meaning and transmit information. Univocal discourse works best when the speaker's code aligns closely with the listener's code. In opposition, *dialogic discourse* serves to generate meaning. Instead of accepting utterances as fixed messages to be stored, as in univocal discourse, utterances take the form of "thinking devices." Utterances are open to negotiation and transformation. In essence, through dialogic conversation interlocutors create mathematical meaning and understanding for themselves and others.

In traditional mathematics classrooms, authority is largely in the hands of the teacher and the text. Communication is mostly one-way with teachers passing on information through lecture or initiation-reply-evaluation (IRE) formats (Mehan, 1979). In an IRE format, the teacher poses a question for which she has a particular answer in mind. She questions a student, who attempts to give her the response that matches her expectation, and then evaluates the response. The IRE instructional pattern illustrates the need to have a match between the codes of the speaker and listener. Overall, univocal discourse is a typical, and perhaps even a defining characteristic, of traditional mathematics classrooms. Social constructivism informs the current reform movement in mathematics education. Consequently, classrooms envisioned by the reform movement are more dialogic. The teacher and students share in the responsibility of constructing mathematical understanding and generating mathematical meaning through give-and-take styles of discourse. While clear communication necessitates both univocal and dialogic elements, Knuth and Peressini have found that mathematics classroom discourse tends to be predominantly one or the other. They call for a need to balance the two approaches by increasing the usage of dialogic discourse.

Teacher Discourse Moves

Krussel, Edwards, and Springer (2004) call the deliberate actions taken by a teacher to mediate, participate in, or influence the discourse in mathematics classrooms *teacher discourse moves.* The teacher has the responsibility to manage the content (topic, task) and structure (small group, whole class) of mathematical discourse, and the physical and temporal boundaries (tools and time) in which it occurs. A teacher's discourse moves can include verbal cues, questions, hints, or invitations. They can also be non-verbal gestures, facial expressions, or the amount of wait time given. Regardless of the form they take, teacher discourse moves have consequences—intended and unintended, short-term and long-term, cognitive, and affective.

By taking into account four elements, the framework offered by Krussel, Edwards and Springer (2004) provides "a structure for considering how the form of a discourse move and the setting in which it is made interplay in determining its consequences" (p. 307). The first element to consider is the *purpose* of the discourse move. There exists a whole range of discourse moves that reflect a teacher's purpose. These include the teacher's intentions to set structural boundaries on the discourse, change the focus of the discourse, build classroom norms for discourse, encourage discourse activity related to reflection or justification, change from a small to whole group discourse structure, and influence participation in discourse. Second, teacher discourse moves relate to the *setting* in which they occur. The physical layout of the classroom is important, as are the time and tools that are available. The setting also includes the sociomathematical norms of the classroom, or standards for explanation and justification, that may evolve over time with instruction. Third, teacher discourse moves are either verbal or nonverbal in *form.* Verbal moves may take the shape of a challenge, probe, request for clarification, request for elaboration, request for participation, invitation for attention, piece of information, hint, or direction. Nonverbal moves include facial expressions, hand gestures, body language, wait time following a question, or changing proximity. Finally, teacher discourse moves result in *consequences.* Sometimes, unintentionally the discourse move may have the effect of lowering the cognitive level of a mathematical task. Discourse moves may also shift a student's attention to a misconception in a concept image or affect the growth of classroom norms. The consequences of discourse moves can be short term, as in influencing the discussion at hand, or long term, as in influencing future patterns of classroom discourse.

Levels and Components of Math Talk Learning Communities

The premises of the review of the literature to this point have important implications for the classroom. To review, the idea that knowledge is transferable has generally been rejected, in favor of a position that knowledge is constructed, both cognitively and socially. By adopting a commognitive framework, we circumvent any inconsistencies, which might come from this dual consideration. As the focus shifts to thinking as communicating, discourse takes center stage. Effective discourse and effective thinking are one and the same. Practically speaking then, we are interested in what constitutes effective classroom discourse—where knowledge is constructed rather than conveyed.

The research on fostering mathematical discourse in the classroom suggests that a redefinition of roles on the part of both teachers and students is in order (Hufferd-Ackles, Fuson & Sherin, 2004; Van Zoest & Enyart, 1998). At the basis of the role shift is a dissemination of power. To have a student-centered classroom, teachers must relinquish to the students some authority. This power sharing transforms the classroom environment, increasing student motivation and engagement while decreasing the adversarial students-versus-teacher relationship (Weimer, 2002). Hufferd-Ackles et al. offer four trajectories for the development of a math-talk learning community in which, "individuals assist one another's learning of mathematics by engaging in meaningful mathematical discourse" (p. 81). The redistribution of power that occurs in a classroom characterized by dialogic discourse follows four trajectories: a) questioning, b) explaining mathematical thinking, c) sources of mathematical ideas, and d) responsibility for learning. Classroom growth, within each of these trajectories can further be categorized in levels, ranging from Level 0 (traditional teacher-directed classroom) to Level 3 (thriving math-talk learning community). Briefly, in a Level 3 classroom students will listen to and question each other's responses independent of teacher initiation; ground their explanations in mathematics rather than social cues; and interject their own ideas and compare, contrast, and build upon each other's ideas. Next we examine the four trajectories in detail.

Who Asks the Questions? What is Their Focus?

There is a long history of studying the art of questioning in educational research (e.g., Cotton, 1989; Dillon, 1990). Questioning is a crucial piece in building a flourishing mathematics discourse community. Fine-tuning two aspects of questioning is especially important in this development. It is necessary to examine *who* asks the question and then to determine the *focus* of the question. In traditional classrooms, teachers predominantly ask students for answers to problems. These short and frequent questions typically elicit brief replies, which the instructor deems correct or incorrect. For teachers using dialogic discourse however, the focus of teacher questioning shifts from students' answers to students' thinking. The teacher will often use follow-up questions to understand student method. Even as teachers progress closer to dialogic discourse by asking probing and open questions, and inviting students to question each other's work, they still assume the primary responsibility for question asking. A defining feature of a Level 3 classroom is the expectation that students will listen to and question each other's responses independent of teacher initiation (Hufferd-Ackles et al., 2004). Power shifts occur in ways both obvious and not so obvious along this trajectory. Clearly, students gain a measure of authority when they become active questioners. More subtly, though, when the focus of the question is on student understanding, they share in the social construction of mathematical ideas. In this way, they exert power over their mathematical knowing and that of others.

What Counts as an Acceptable Mathematical Explanation?

Another powerful tool for improving classroom discourse is to consider the nature of mathematical explanation. In classrooms dominated by univocal discourse, the instructor rarely elicits any kind of mathematical explanation. As teachers become increasingly adept at facilitating mathematical discourse, they move from asking for one or two solution strategies to asking for multiple solution strategies. Ultimately, in a Level 3 classroom rich in dialogic discourse, instructors probe, stimulating their students to deeper understanding through their own mathematical explanations (Hufferd-Ackles et al., 2004).

It is also valuable for instructors to recognize what counts as an acceptable mathematical explanation in their classroom, taking into account both their perspective and that of their students. The standard that students and teachers come to deem acceptable for mathematical explanation is an example of a sociomathematical norm (Yackel & Cobb, 1996). There is an important distinction between social explanations and mathematical explanations. Promoting an environment where students ground their explanations in mathematics rather than being motivated by social cues (i.e., providing the explanation the student thinks the instructor wants to hear or one that is socially acceptable by his or her peers) is significant for dialogic discourse.

Who has Control Over the Mathematical Ideas?

The locus of control over mathematical ideas is another important factor for instructors to consider in fostering mathematical discourse. In a traditional classroom, the teacher projects him/herself as the sole purveyor of mathematical ideas. In contrast, in a classroom characterized by dialogic discourse, the power structure becomes more diffuse. Interjection of student ideas is common and students compare, contrast, and build upon each other's ideas of their own volition. At this developmental stage, the teacher retains the responsibility for making choices about the direction of the class. However, unlike a traditional classroom in which a teacher adheres to a rigid instructional plan, teachers promoting dialogic discourse may modify or extend instruction based on student ideas (Hufferd-Ackles et al., 2004).

Speaking of the challenges that students face trying to learn an academic discourse, and in so doing highlighting the difference between univocal (convey meaning) and dialogic (generate meaning) discourse, Northedge (2003) states:

> Explanation generally achieves more in the mind of the teacher, where it is being actively generated, than in the minds of students. A strong flow of debate is much more likely to enable new knowing, particularly with a diverse student body. (p. 177)

Northedge's observation illustrates that to learn something, one must grapple with the ideas. When teachers invite students to share in classroom discourse, thereby giving students ownership of mathematical ideas, they enable new mathematical knowing and understanding.

Who has the Responsibility for Learning?

Characteristics of a classroom dominated by univocal discourse include the teacher repeating student responses for the entire class and acting as the primary arbiter of correct answers. This typically results in low student responsibility for learning. As classrooms exhibit increasingly sophisticated dialogic discourse, students help one another and agree or disagree with mathematical ideas. Ultimately, students serve as co-evaluators of mathematical ideas and are responsible for clarifying their own understanding as well as that of their classmates (Hufferd-Ackles, Fuson & Sherin, 2004).

By being mindful of who asks the question, what counts as a mathematical explanation, and who has control over mathematical ideas in their classrooms, instructors can purposefully choose discourse moves to increase student responsibility for learning. Moreover, Tsui (2002) showed that actions/discourse moves that increase student responsibility for learning may also result in students experiencing greater growth in critical thinking (students abilities' to identify issues and assumptions, recognize important relationships, make correct inferences, evaluate evidence or authority, and deduce conclusions). These discourse moves include:

> having both professors and students ask more questions in class; encouraging students to respond to questions posed by their peers; seeking not only a greater degree of discussion per se, but participation by a greater proportion of students; motivating students to question or challenge what is being said; complimenting students on their contributions to the discussion; and encouraging students to volunteer comments rather than participating in discussion only when they are called upon or have a question. (p. 758)

Ultimately, through dialogic friendly discourse moves related to questioning, explanations, and mathematical ideas, instructors send an empowering message to students, one of trust and confidence in the students' capability to take responsibility of their own learning.

Mathematical Discourse within the Framework of Commognition

Discourse as a Type of Communication

Communication was previously defined as a collective activity that follows patterned actions and re-actions that evolve across time within a community. Accordingly, there are a multiplicity of communications, each one made distinct by its rules, objects, and types of mediators. Sfard (2008) defines discourse then as "the different types of communication, and thus of commognition, that draw some individuals together while excluding some others" (p. 91). Naturally, it follows that within human society there exist a multitude of overlapping *communities of discourse.* Having rejected a more traditional notion of communication, one in which individuals exchange information; face-to-face interaction is not a criterion of membership in a discourse community. Rather "membership in the wider community of discourse is won through participation in communicational activities of any collective that practices this discourse, however small this collective may be" (p. 91).

Mathematics as Discourse

One way to delineate between discourses is to identify their objects. The object of discourse in zoology, for example, is the animal. Similarly, mathematics is a discourse about mathematical objects—numbers, functions, sets, and geometrical shapes.

However, unlike other disciplines where the objects are concrete, mathematics is a discourse about abstract objects.

> *Mathematics begins where the tangible real-life objects end and where reflection on our own discourse about these objects begins.* Indeed, mathematical discourse, especially when frozen in the form of a written text, can be seen as a multi-level structure, any layer of which may give rise to, and become the object of, yet another discursive stratum. From this description, mathematics emerges as an *autopoietic* system—a system that contains the objects of talk along with the talk itself and that grows incessantly "from inside" when new objects are added one after another. (Sfard, 2008, p. 129)

We cannot physically discover mathematical objects, per se, in the world—they do not precede the talk about them. Mathematical objects are discursive constructs. How then, if not by concrete objects, can we distinguish discourse as mathematical? While identifying one characteristic unique to all discourses mathematical in nature is unrealistic, speaking of "family resemblances" is not. Sfard offers four properties useful for determining whether a case of discourse counts as mathematical. The properties consist of two tools and two procedural forms/outcomes respectively discussed next.

Mathematical Nouns (words)

One defining feature of a discourse is its terminology. Mathematical words most often, but not always, pertain to quantity and shape. In contrast to colloquial discourse, word use in mathematics is highly disciplined. "Mathematical communication. . .more than any other, is likely to be hindered by considerable differences in interlocutors' use of words" (p. 135). The abstract nature of mathematical objects and the varying degrees of objectification by interlocutors amplifies the consequence of word use in mathematical conversations.

Visual Mediators

It has been noted that mathematics is unique in that its objects are largely intangible as opposed to those of other discourses. We can easily scan with our eyes the material objects of our everyday conversations. While we may *represent* mathematically constructed objects using visual means, we can never truly *show* them. Nonetheless, Sfard (2008) argues that what we see is no less important in communication about abstract objects than tangible ones. Visual mediators are "the providers of the images with which discursants identify the object of their talk and coordinate their communication" (p. 147). Visual mediators include both perceptually accessible objects—those that exist independent of discourse—and artifacts such as symbols, diagrams, graphs, and drawings that were created specially for the sake of communication. So what exactly do visual mediators "go between"?

> Mathematical communication involves incessant transitions from signifiers to other entities. . .called *realizations* of the signifiers. *Signifiers* are words or symbols that function as nouns in utterances of discourse participants, whereas the term *realization of a signifier S* . . .is a perceptually accessible thing S' so that every endorsed narrative about S can be translated according to well defined rules into an endorsed narrative about S'. (p. 154)

The subjects of endorsed narratives and rules of discourse are taken up next.

Endorsed Narratives

The means by which a discourse community renders truth is yet another marker of its distinction. Within the commognitive framework, truth is packaged in what Sfard (2008) calls an *endorsed narrative.*

> Narrative is any sequence of utterances (communicational acts) framed as a description of objects, of relations between objects, or of processes with or by objects, that is subject to endorsement or rejection with the help of discourse-specific substantiation procedures. (p. 134)

Conditions of endorsement differ widely between discourses. One important consideration is the relation between the power structure of a community and its endorsement of a narrative. Our present day version of historical truth, for example, is different from what it was just a few generations ago, as historians begin to include multi-vocal accounts of history in texts and the halls of education. Mathematical discourse, however, is known as being "impervious to any considerations other than purely deductive relations between narratives" (p.134). Mathematical truth lies in the consensually endorsed narratives of scholars—definitions, proofs, and theorems.

Routines

Human communication is a rule-regulated activity. Sfard argues that mathematical learning is the initiation into the well-defined discourse of mathematics—and hence the rules that govern it. Mathematical discourse is distinguished by object-level and meta-level rules (Sfard, 2008). "Object-level rules are narratives about regularities in the behavior of objects of the discourse, whereas meta-rules define patterns in the activity of the discursants trying to produce and substantiate object-level narratives" (p. 201). An example of an object-level rule for geometry would be: the sum of the measures of the interior angles in a triangle equals $180°$. On the other hand, the rules that guide the act of geometric proof are at the meta-level. It is fascinating to note that while object-level rules are immutable,

> mathematics is an autopoietic system that grows by annexing its own metadiscourses, and this means, among others, that what counts as a meta-rule in one mathematical discourse will give rise to an object-level rule as soon as the present metadiscourse turns into a full-fledged part of the mathematics itself. (p. 201)

Sfard illustrates how a meta-rule in arithmetic discourse (finding the product of the sum of two numbers and a third number is the same as finding the sum of the products of the third number and each of the two addends) becomes an object-level rule in algebraic discourse (i.e., The Distributive Property: a[b+c]=ab + ac).

In addition to evolving over time, metadiscursive rules are tacit, normative, constraining, and contingent (Sfard, 2008). An elaboration of these five characteristics will be instructive in understanding the role that metarules play in regulating the communicative effort. First, rules of communication develop gradually over time. Sfard likens the process to natural selection, where the fittest survive and those that do not work are slowly phased out. In the classroom, the discourse rules are an evolving product of teacher and student interactions. In fact, it might be said that the goal of mathematics education is the ongoing modification of students' discourse so that they can join the historical discourse of the discipline. Second, much like laws of nature, metarules are tacit in the sense that the interlocutors do not have to be purposeful in their attention to them. A rock on the edge of a cliff, for example, drops by the law of gravity. On the other hand, humans, unlike falling rocks, do have the capacity for reflection. And it is actually through this reflection that mathematicians recognize the patterns in their own actions and transform these patterns into new mathematical objects. Thus, "explicating its own metarules is one of the fundamental activities of mathematics" (p. 204). Third, metarules governing mathematical discourse are normative. To be a norm, the rule must be both widely enacted within the mathematical community and endorsed by the majority, especially those deemed experts. Fourth, metarules tend to constrain, rather than determine communication. Instead of dictating communication, they serve to eliminate wrong choices for communication. Finally, metarules are contingent. They are not the result of some external objective reality, but rather subject to historical human judgments and choices.

To gain a greater understanding of mathematical discourse—a communication made special by its distinct patterned activity—we look to uncover the structure of mathematical *routines.* Sfard (2008) defines routines as "sets of metadiscursive rules that describe recurrent discursive patterns" (p. 220). A subdivision of the set of rules helps to define the "how" and "when" of the routine. The three subsets of metadiscursive rules and their function as defined by Sfard are:

- *Applicability conditions* subset of routine-defining metarules, composed of rules that delineate, usually in a nondeterministic way, the circumstances in which the given routine course of action is likely to be undertaken by the person. (p. 295)

- *Routine course of action (or routine procedure)* set of metarules that determine (e.g., in numerical calculations) or just constrain (e.g., in proving or writing a poem) the way the routine sequence of actions can be executed. (p. 302)
- *Closing conditions (closure)* defining circumstances that the performer is likely to interpret as signaling a successful completion of performance. (p. 296)

The "how" of routine, found in its course of action, is often the more straightforward of the tasks. Knowing "when" to apply a routine and "when" it is satisfactorily complete, however, tend to prove more elusive. The hallmark of mathematical creativity is in the "when"—in applying familiar rules in new circumstances.

Learning Communities

Purposeful Community

Near the end of the 20th century, the Carnegie Foundation for the Advancement of Teaching (1990) undertook a year-long project researching community on college campuses. The project was motivated by frustrations stemming from the darker side of campus life—drunkenness, incivility, sexual harassment, and racial harassment—along with the desire of college administrators to reinvigorate their institutions with intellectual and social vitality. Since the publishing of the report, which highlighted six principles to guide the kind of community that every college and university should aspire to, much has been written about the "powerful potential of learning communities" (Lenning & Ebbers, 1999). While these contributions are indeed helpful, the centrality of the first principle of the Carnegie report is of utmost interest for this study.

The first principle, which is fundamental to all others, states "A college or university is an educationally *purposeful* community, a place where faculty and students share academic goals and work together to strengthen teaching and learning on the campus" (Carnegie Foundation, 1990, p.9). In essence, this principle says that the intellectual pursuit must be the community's prime focus. Without this, the community crumbles and the remaining principles are mere afterthoughts. The report makes several recommendations for creating a purposeful common intellectual commitment. Small seminars, for example "are needed so that undergraduates can have more direct access to professors in a setting where dialogues thrive and relationships grow, not just between teachers and students, but among the students themselves" (p. 12). It is in these small settings where students have an opportunity to learn to cooperate and not just compete. The report also highlights the natural and fitting role that academic departments have in creating purposeful community.

> Beyond the classroom, community can be strengthened by academic departments that bring students and faculty together. The department is, perhaps, the most familiar, most widely accepted organizational unit on campus. As students select a major, they join with faculty to pursue common academic interests and often forge social loyalties, too. In addition to their advising role, departments can become a creative intellectual and social unit on the campus through special seminars, lectures, and social events for students and faculty. (p. 13)

In their academic departments, colleges and universities have a built-in infrastructure that they can exploit and broaden in their quest of community. In summary, most fundamental in the pursuit of community on college campuses is a relentless focus on the academic purpose at hand.

Formation of Learning Communities

Several practicalities of the learning community, peripheral of course to its purpose, are next discussed. First, learning communities can take on many forms. They encompass "a variety of curricular approaches that intentionally link or cluster two or more courses often around an interdisciplinary theme or problem, and enroll a common cohort of students" (Smith, MacGregor, Matthews & Gabelnick, 2004, p. 67). At the core of all learning communities is the goal to connect people and ideas. Palmer (1998) suggests a metaphorical model for the formation of learning communities. In the model, the subject material is at the center with students and teacher circling around it and learning together. Wilson, Ludwig-Hardman, Thornam and Dunlap (2004) identify a bounded learning community as one which forms in a structured versus spontaneous setting, such as a college course. They contend that the teacher's role is critical in bounded learning communities. In particular, they suggest that teachers need to provide the infrastructure for interaction and work while modeling effective collaboration and knowledge construction.

A *de facto* formation of community among upper division students within the same discipline is common on a college campus. These students often share the same courses within their discipline and have begun to adopt a professional identity. Similar to the Carnegie Foundation, Lenning and Ebbers (1999) suggest the potential value in forming curricular area learning communities. "If faculty in a disciplinary department intentionally organize their student majors into meaningful discussion and study groups that collaboratively facilitate learning and commitment to the values of the discipline, those student groups can become well-defined and effective learning communities" (p.

27). The use of seminars and colloquiums is one way to facilitate this organization in the mathematical discipline. Fleron and Hotchkiss (2001) for example, report on the success of first-year and senior seminars that focus on reading mathematics, writing in mathematics, and proof. The seminars provide mathematics students with access to the culture of mathematics and the broader mathematical experience. Similarly, Brabenec (2001) describes the value of an informal colloquium setting in which sophomores are introduced to mathematical exploration, research, problem-solving, group work, and oral presentation.

Another important area, where one can see strains of the value of fostering purposeful learning communities, is the research on recruiting and retaining minorities in the Science, Technology, Engineering, and Mathematics (STEM) fields . May and Chubin (2003), for example, identify key elements of intervention programs designed to retain minority engineering students. Inherent to successful intervention programs is an element of collaborative learning. Thus, components of successful programs often include a thorough and year long freshman orientation, clustering—wherein students are enrolled in the same courses, the availability of a physical location or study center for students to gather, and structured study groups. Improved academic performance, improved retention, improved oral communication skills, increased student satisfaction of the learning experience, and higher student self-esteem are all results of intervention programs that focus on collaborative learning. The importance of collaboration reported by May and Chubin corroborates with findings on minority persistence in STEM related disciplines. Grady (1998), for example, found that the support students gained from other minorities had a very important effect on science ambition and commitment to science during their sophomore year. Walters (1997) also reported on the importance of social integration through involvement with peers, faculty, and campus activities, as an important factor in reinforcing student persistence.

Learning Mathematical Proof

Proof is at the heart of the mathematical disciplines. Put simply, it is the logical organization of the irrefutable evidence that a theorem is true (Hale, 2003). A proof outline consists of a hypothesis connected to a conclusion through a series of statements justified by axioms, theorems, definitions, previous steps in the argument, and rules of logic (Greenberg, 1993). While a proof itself is the quintessence of deductive reasoning, the discovery of proof is a largely creative and inductive process.

Recent research in mathematics education highlights the ways in which mathematicians engage in their discipline. Sriraman (2004), for example, studied the processes employed by mathematicians to seek insight into mathematical creativity. The results of the study indicate that the four-stage Gestalt model of preparation-incubation-illumination-verification provides a framework for understanding mathematical creativity. Additionally, Sriraman found that social interaction, imagery, heuristics, intuition, and proof are all contributing characteristics of the construct. Two observations bear mentioning for the purposes of the proposed study. First, "all of the mathematicians [in Sriraman's study] acknowledged the role of social interaction in general as an important aspect that stimulated creative work" (p. 26). Further, mathematics classrooms and curricula rarely allow students to engage in mathematics in the way that mathematicians do. To capture the complexity that is creative mathematical thinking, mathematics classrooms should afford students opportunities for prolonged periods of engagement with the material as well as the independence to formulate solutions.

Logical inference forms the basis of formal proof. Weber (2008), though, found that mathematicians also rely on informal deductive reasoning, example-based reasoning, and conceptual knowledge for proof validation. What is more, the social context has an influence on proof validation. Specifically, mathematicians tend to vary their validation practices according to the a) person authoring the proof (e.g., student versus mathematician) and b) mathematical domain in which the proof is situated. In general, the mathematicians were more willing to devote time and energy to analyzing a suspicious proof argument if they trusted the proof's author. Additionally, mathematicians in the study judged a theorem on not only content, but also its status within a mathematical community. In other words, they considered whether a theorem was established or generally accepted in the mathematical domain under question. "If students were aware of the need for making the latter decision [between proof for large mathematical community versus proof for situated community]," Weber suggests, "they may come to appreciate the social functions of proof in helping mathematical communities understand why certain theorems are true" (p. 452).

Discovering proof is a complex process, which students often struggle to learn. Moore (1994) identified three major and interrelated sources of cognitive difficulties that mathematics and mathematics education majors have in learning proof. They are a) concept understanding, b) mathematical language, and c) getting started on a proof. Student difficulty in getting started on a proof is an amalgamation of their lack of understanding and use of language and notation and their deficiencies in conceptual understanding. Moore found, for example, that students often needed concept images such as examples, diagrams, and graphs for knowledge and understanding of formal definitions. In turn, the students' knowledge of definitions influenced their ability to make use of them in the proof process. Thus, as students become more comfortable with notation, mathematical grammar and syntax, and the logical structure of proofs, they may rely less on informal concept images. Perhaps most interesting, is the difference in cognitive structures for proof that Moore identified between students and instructor. While students maintained separate schema for mental images, concept definitions, and procedures for using definitions, the professor incorporated all of this knowledge into one fluid schema.

Weber's (2001) study of undergraduate ability to construct proofs in abstract algebra reveals that an accurate conception of what constitutes proof and an adequate syntactic knowledge base might not be sufficient for proof construction. The principal difficulty of students with proof in this study was a lack of strategic knowledge. Weber suggests that individuals may not gain strategic knowledge through experience alone and that there is a need for more research on how students acquire effective strategic knowledge.

The largely inductive, intuitive, and non-linear nature of the process of proof makes it difficult to teach (Hale, 2003). Nonetheless, students need support to bridge the gap from high school and lower division undergraduate mathematics courses to courses that are more theoretical in nature. Thus, many institutions offer a transition-to-proof course. These courses often introduce students to logic and proof methods such as direct, contrapositive, contradiction, and proof by cases. Baker and Campbell (2004) suggest, however, that a heavy emphasis on logic within these courses may result in a somewhat flawed student understanding of proof as a problem with a solution. This finding builds on Moore's (1994) finding of undergraduates' cognitive difficulties with proof:

> The concept image of some students was that of proof as explanation, whereas for others proof was a procedure, a sequence of steps that one performs. It was not clear to what extent students viewed proof as a piece of mathematical knowledge, an object. (p. 264)

As such, Baker and Campbell recommend activities for transition-to-proof courses that encourage students to appreciate not only the concluding step of a proof, but also the insight that it offers into mathematics.

Dean (1996) offers a six-step Polya-like model for teaching the proof process. Dean compares the process to a chess game in which opening and closing moves are routine, with creative play occurring in the middle. In the first or *opening* phase of the model, like the opening move in chess, students gain insight into the theorem through careful reading. Students should list everything that they know about the theorem as well as the unknown. They might also translate a statement in symbols to words. During the second phase, *brainstorm*, students play with all ideas that come to mind, without regard for critical analysis. Guessing and checking is more than appropriate, as is leaving the theorem only to return to it later during this stage. The third phase, *instantiate,* is the "Aha" moment when the student recognizes a path that links hypothesis with conclusion. A critical examination of the argument is required during the fourth stage, *convince*, to see if the argument is indeed valid. Students are encouraged to take a devil's advocate like stance toward their argument at this time to ensure that there are no gaps in the reasoning. The fifth stage, *reflect*, is a powerful opportunity for students to learn from their proof. It is an opportunity for students to consider questions such as:

- Could I have proven this theorem differently?
- Even though I established this theorem independent of other theorems, could I have taken advantage of them in establishing this proof?
- What did I do that caused me to see through the maze to solve this?
- What strategies did I use to get this proof?
- Could I skip some of these inferences and still have a valid proof?

In the sixth and final stage of the proof process, *extend*, the students attempt to apply what they have learned in one mathematical system (e.g., 2-D plane), to another system (e.g., 3-D space). They can additionally consider new properties that are a result of the proved theorem.

Dean's (1996) six-phase model is significant for introducing students to mathematical proof. First, Dean's model de-emphasizes the algorithmic process, one that Baker and Campbell (2004) suggest is detrimental to undergraduate mathematics majors' understanding of proof. Secondly, the model coincides with the Gestalt model (preparation-incubation-illumination-verification) that Sriraman (2004) associated with mathematical creativity. And finally, the final two stages of Dean's model, reflection and extension, lend themselves well to dialogic mathematical discourse.

Finally, let us consider what it means to "learn mathematical proof" from a commognitive perspective. To review, to learn mathematics is to become fluent in its discourse—one recognized as mathematical by experts in the discipline. This fluency requires a command of the discourse's objects as well as its meta-discursive rules.

Moreover, mathematical proof is the vehicle by which the mathematical discourse community determines truth. Thus, to substantiate truth, "one produces a proof—a sequence of endorsed narratives, each of which is deductively inferred from previous ones and the last of which is the narrative that is being endorsed" (Sfard, 2008, p. 232). We are to understand the endorsement as one that is consensual by the mathematics community.

Summary

This literature review has aimed to provide a backdrop for exploring the role that mathematical discourse plays in undergraduate mathematics majors' learning of mathematical proof and in a learning community. The notion that knowledge is constructed rather than transmitted is foundational to the discussion. Furthermore, acknowledging the role that social interaction plays in the construction of knowledge—regardless of whether that construction occurs "in the head" or externally—is of prime importance. Then, the adoption of the theoretical framework of commognition, where thinking and communication are analogous, lays the groundwork for studying the interrelatedness of discourse, proof, and community.

CHAPTER 3

METHODOLOGY

This chapter outlines the research methods for studying undergraduate mathematics majors' learning of mathematical proof and their sense of community in a seminar utilizing small group discourse. The first section of this chapter provides a brief discussion on the qualities of qualitative research. Then a section is devoted to introducing the commognitive approach to research and explaining how it fits within a qualitative paradigm. This leads to the justification of the research methods. Additional sections address the topics of subjects, setting, principal investigator's role, data collection and instrumentation, and data analysis.

Qualitative Research

Qualities of Qualitative Research

Qualitative research is more than a set of techniques and procedures. It is in a larger sense an *approach* to inquiry. At its core, qualitative research is a quest to understand the "meanings individuals construct in order to participate in their social lives" (Hatch, 2002, p. 9). Wiersma (2000) offers, as one of five epistemological underpinnings of the qualitative approach, the following:

> It is the perceptions of those being studied that are important, and, to the extent possible, these perceptions are to be captured in order to obtain an accurate "measure" of reality. "Meaning" is perceived or experienced by those being studied, it is not imposed by the researcher. (p. 198)

Hatch expands on the same point—on the centrality of meaning in qualitative research. He traces the philosophical roots of qualitative research back to the interpretive sociology of Max Weber. Weber worked to develop a research methodology that would set the cultural sciences apart from the natural sciences. As a means "to reach causal explanation of and general laws about patterns in human behavior," Weber stressed "understanding (*verstehen*)" or "interpretation" (Rosenberg, 1983, p. 50). His methodological stance, then, was one of explanatory understanding. Modern interpretive sociologists take meaningful interaction between persons as their object of analysis. They "attempt to grasp and describe the richness of meanings used by the participants in the situation under investigation and to explain the action observed in terms of these meanings" (p. 59).

There are other qualities, in addition to but interrelated with the centrality of meaning, fundamental to the qualitative approach. The qualitative researcher, for example, views the phenomena under investigation holistically, with an eye towards complexity (Hatch, 2002; Wiersma, 2000). As a result, "qualitative reports are usually complex, detailed narratives that include the voices of the participants being studied" (Hatch, p. 9). Furthermore, the qualitative researcher is interested in the lived experiences of subjects. Out of this concern for context, qualitative researchers carry out their inquiry with openness about what they will observe in the natural setting. Consequently, qualitative research designs tend to be flexible, evolving, and emergent (Hatch, Wiersma).

Commognitive Research

From the commognitive perspective, learning is the social phenomenon of participating in the communicational activities of a distinct community. In the spirit of

Weber, the researcher seeks understanding (*verstehen*) or explanatory interpretation of this highly complex human activity. Sfard (2008) explains:

> The quest for discursive patterns is the gist of commognitive research. Repetitions may be occurring in different aspects of discourse and across different fields and ranges. Sometimes, we are searching for what stays invariant across the whole community, and sometimes we scrutinize only the discourses of newcomers. On other occasions, we search for patterns typical of mathematical discourses in schools, and in yet other cases we satisfy our selves with what remains constant over time in the mathematical discourse of a certain classroom or even just in the discourse of an individual student. As in any other type of research, familiarity with what stays the same through incessant change is the basis for our understanding of phenomena and for our ability to extrapolate beyond the present set of data into a range of future situations. (p. 200)

Sfard (2002) is careful to point out, however, that the best researchers can hope for is a "*convincing interpretation*" that is "as compelling, cogent and trustworthy as possible" (p. 32). Furthermore, we must regard the resulting interpretation as one of many, as a tentative and incomplete product.

While the goals and interpretive stance of commognitive investigation are fairly well defined, efforts to build a strong research methodology to support this research framework are still developing. Nonetheless, Sfard (2002) states:

> It is clear that the proposed conceptualization of thinking implies a wide range of data-collecting strategies and can be expected to produce a rich and great diversified family of analytical methods. In addition to the already existing

> discourse and conversation analyses, those who work within the communicational approach to cognition have yet to construct and test their own methods of handling data, tailored according to their specific need. (p. 31)

Despite a fully-fleshed out methodology, the commognitive approach to research is complementary if not compatible with the qualitative paradigm. Table 1 compares commonly agreed upon characteristics of qualitative research with the goals and characteristics of a commognitive approach to research.

Justification of Qualitative Approach

This study sought insight into an inherently social phenomena—the learning of mathematical proof by undergraduate mathematics majors in a seminar utilizing a small-group format. As such, data collection and data analysis were primarily qualitative in nature, allowing for deep understanding of the phenomena. The methodology does not allow the findings to be separated from the context. However, as is true of qualitative research, the value of the research lies not in generalizing to a larger population, but in the "immediate implications for action of knowledge framed in interpretive terms" that result from "close and direct observation of the micro context of interaction and detailed interviews with the people engaged in it" (Rosenberg, 1983, p. 60). As Patton explained in a 1985 invited address to the American Educational Research Association (as cited in Merriam, 2002), qualitative research does not endeavor to predict the future. Instead, it seeks, as the end itself, an understanding of the nature of the setting.

Table 1

Comparison of Qualitative and Commognitive Research Characteristics

Qualitative Characteristics[1]	*Commognitive Characteristics*
Inductive Inquiry	The commognitive researcher searches for patterns in observed instances to "extrapolate beyond the present set of data into a range of future situations"[2]
Understanding Social Phenomena[3]	The unit of analysis in commognitive research is discourse—a special type of communication (*col*lective activity) that follows patterned actions and re-actions that evolve across time within a *com*munity. (Note the prefixes *com-* and *col-* mean "with"/ "jointly"). Participation in discourse is viewed as a communal/social activity.
Atheoretical	Commognitive research is less about generating theory and more about identifying and describing discursive patterns.
Holistic Inquiry	Commognitive research assumes holistic interpretation and avoids exclusivity in its claims.
Context-Specific	Commogntive researchers are interested in discourse defined by the context of the community in which it occurs (e.g., classroom, school, academia)
Observer-Participant	The commognitive researcher is a participant and observer.[4] In the work of Sfard and Kieran (2001), for example, the authors were "circulating in the class (research setting), helping the students and observing" (p. 43). Their research assistants helped to teach the class.
Narrative Description	The results of commognitive research are nonstatistical. They are instead narrative descriptions of the circumstances and evolution of discourse. See Kieran (2002) for an excellent example.

[1]Wiesrma (2000, p. 13); [2] Sfard (2008, p. 200); [3] Wiersma (2000) says "Qualitative research is done for the purpose of understanding social phenomena, *social* being used in a broad sense" (p. 13); [4]Sfard (2008, p. 281)

There exist multiple perspectives on the nature of mathematics and the related nature of mathematics education. This multiplicity, into which Chapter 2 provided some insight, spills over into the aims of mathematics education research. Teppo (1998) states that "the incorporation of qualitative methodologies into mathematics education research has made it possible to investigate the teaching and learning of mathematics at new and different levels of complexity and from multiple perspectives" (p. 10). In particular, qualitative research has aided in the development of explanatory models of what constitutes mathematical learning. Moreover, it has expanded conceptions of what is possible in mathematics education. The "diverse ways of knowing," or varied approaches to mathematics education research, make it a vibrant field. "Diverse ways of knowing" also necessitate openness and communication within the field. Accordingly, the remaining sections of this chapter attempt to make the details of the methodology employed in this study transparent.

Participants

Participants in the study were freshman and sophomore mathematics majors enrolled in a zero-credit seminar at a small Western Pennsylvania university during the fall 2008 semester. All students enrolled in the seminar received an invitation to participate. Participation in the study was voluntary. A total of nine students—two males and seven females—agreed to participate in the study. Four of them were freshman and five were sophomores. Furthermore, five of the students (all female) were working towards a secondary education concentration in addition to a bachelor's degree in mathematics. At the time of the study, all participants except one were enrolled in the university's calculus sequence; hence no participant had completed the calculus sequence. Four of the sophomore participants had taken a discrete mathematics course the prior spring semester. Discrete mathematics and calculus comprised the upper limit of participants' college mathematics course taking.

Five students attending the seminar declined to participate in the study. We can surmise three reasons why this was the case. First, four of the five non-participants had to miss part of or the entire seminar on a somewhat regular basis because of conflicting commitments such as athletics and work. These students may not have felt comfortable participating in the study due to attendance concerns. Second, the audio recording portion of the study may have dissuaded some students from participating. It seems possible that some students fear having gaps in their understanding exposed by audio recording. Third, a lack of first-hand knowledge about the research process may have intimidated students from participation. This too seems plausible, given that the seminar was for lower division students. The study was primarily qualitative in nature. Thus, while it is important to discuss potential factors of non-participation, this should not undermine the rich insights gained from those who did participate.

Setting

The study took place at a rural Western Pennsylvania university. The university enrollment included 1,553 undergraduate and 572 graduate students during 2007-2008 academic school year. Females and males comprised 64.1 % and 35.9% of the student body respectively. The university aims to offer higher education in an environment guided by Catholic values and teachings.

The zero-credit seminars were sanctioned by the university curriculum committee in spring 2007, and carried a course number with a Math prefix. However, the seminars carry no letter grade. The university's Office of Advising and Retention included the seminar on all incoming freshman mathematics majors' schedules for fall 2008. Rising sophomores were encouraged by their advisors to include the seminar on their fall 2008 schedules during the registration period in spring 2008. The seminar was implemented for the first time during the fall of 2008. It convened weekly, ten times, with each meeting lasting 75 minutes. Data collection for the study occurred during six seminar meetings. The interview portion of the study took place in the student lounge of the student union building on the university's campus during the first week of the spring 2009 semester.

Principal Investigator's Role

Qualitative research requires that the researcher make known any bias. Accordingly, I am a faculty member at the university where the study took place. I conducted the seminar and study while on a leave of absence during the 2008-2009 academic year. The seminar was offered for the first time during fall 2008 and I had complete academic freedom over its design. I was a proponent of initiating the seminars in an effort to build community and raise standards within the mathematics program. Thus, I had a stake in the success of the seminar.

In the study, I assumed the role of participant-observer, a common role for commognitive researchers (Sfard, 2008). In leading the seminar, I attempted to facilitate mathematical discourse among the student participants. As such, I refrained from lecturing and utilized a small-group format. In this setting, students and the instructor share the responsibility for mathematical questioning, explanations, ideas, and learning. The design of this particular study, then, places my role as the investigator on the spectrum between the designations of active participant and privileged observer. On the one hand, I participated alongside students as they engaged in mathematical discourse about mathematical proof. On the other hand, I had the unique responsibility of providing students with a framework for discourse as well as monitoring the direction of that discourse.

Action research is a type of applied research centered on the investigation and improvement of practice (Hatch, 2002). It is distinct from other research methods in that those involved in the collection and analysis of data may simultaneously belong to the group under study. Fundamental to dual involvement is the desire of the researcher to use the research findings to bring about an element of change (Bogdan & Biklen, 1992; Hatch, 2002; Patton, 2002). "There is recognition," Hatch states, "that the values of the researcher have a prominent place in the inquiry, and change is the desired endpoint" (p. 31). Action research generally follows a pattern of problem identification, planning for change, implementing change, and assessing effectiveness of change.

The design methodology for the study under discussion borrowed two key features from action research. First, the implementation of the zero-credit mathematics seminar was an attempt to address and improve upon three areas of the university's mathematics department. These included a notable lack of a sense of community and collaboration among mathematics majors, a lack of support for students making the transition from high school/lower division courses (calculation based) to upper division (proof based) mathematics courses, and a concerted effort to retain/recruit mathematics majors. Second, as principal investigator, I was both a proponent of and instrumental in the initiation of the seminar. Thus, I taught the seminar and will use the data to improve practices within the department.

Because the primary goal of action research is often to improve practice or solve a problem in a local context, research methods and procedures tend to be more informal than methods employed by scholars looking to develop a theory or contribute to a disciplinary knowledge base. While I chose a research problem rooted in my own professional practice, I employed rigorous research methods from the commognitive research field as well as from established qualitative research traditions. These methods and procedures for data collection and data analysis are described in the next sections.

In summary, decisions about the level of participation of the researcher in qualitative research inevitably have consequences. One advantage is that participants did not react to the investigator as a "stranger" in the room. The decision, however, does have implications for the objectivity of the results. The trustworthiness of the study is discussed just prior to the conclusion of the chapter.

Data Collection: Instrumentation and Procedures

Data Collected

The first part of the research methodology concerns data collection. In basic interpretive qualitative study, data is collected through interviews, observations, and document analysis (Merriam, 2002). This study included variations of all three forms of data collection (see Table 2). The data collection of this study shares similarities with those used by Kieran (2002). Specifically, there was an attempt to balance data that represents both the collective and individual activities of students. Data collection, for example, includes individual interviews on the learning that occurs through collective interaction. The multiple data collection methods of the study increase its trustworthiness—a topic taken up later on in the chapter.

Table 2

Data Collection Forms

Basic Interpretive Qualitative Study	*Current Study*
Interviews	Individual interviews were conducted with participants approximately 20 minutes in length during the first week of spring 2009 semester.
Observation	The phenomenon under investigation is student learning of mathematical proof. Since learning is defined as increasingly expert participation in discourse, the unit of analysis (and that which was being observed or *noticed)* was the discourse between learners. Digital audio recordings of small-group and whole-group discourse were collected over the course of six seminar sessions. The investigator also took field notes for each seminar session.
Documents	Participants' work on mathematical proof was collected from all six seminar sessions. In addition, the investigator collected pre- and post-seminar responses and survey data from participants.

Data Collection Procedures

Students were asked to complete three pre-seminar readings over the course of six sessions from Daniel Solow's (2005) *How to Read and Do Proofs.* The readings ranged from six to nine pages in length. At the commencement of each seminar following a reading assignment, students were asked to respond in writing to the following two open-ended questions.

- What is the main understanding that you gained from the reading?
- What questions/confusion/curiosities/misunderstandings do you have from the reading?

This opening instructional method was similar to that employed by King (2001). In his lecture-free seminar courses, he used a structure that includes pre-seminar readings and exercises as well as pre-seminar reaction pieces. Participants in the study under discussion were not asked to complete exercises prior to the weekly seminar meeting. I refrained from lecture and instead worked to foster a more dialogic discourse among students, especially using small-group discourse. Exercises in the Solow text served as the basis for introductory whole-group seminar discussion and subsequent small-group work. Throughout the seminar, the brief whole-group and more lengthy small-group discussions were audio-recorded. During small-group work, participants were encouraged to collaborate in constructing proofs, express their ideas aloud, and assist each other in understanding the proof constructions.

At the conclusion of each seminar session, participants were asked to respond in writing to the following two additional open-ended questions.

- What mathematical insights/understandings/connections did you have in seminar today?
- What specifically contributed to these insights/understandings/connections (e.g., specific example, comment, or question posed?)

Immediately after each session ended, with the assistance of the audio recording, I engaged in writing field notes. Bogdan and Biklen (1992) define fieldnotes as the written notes of what the researcher hears, sees, and experiences in the course of collecting and reflecting on the data. Fieldnotes serve a dualistic purpose. They are simultaneously descriptive and reflective. The descriptive portion of field notes contain the investigator's best objective record of what occurred in the seminar setting. I made every effort to capture all detail relevant to the study's purpose. Furthermore, I wrote reflective notes. The reflective notes took account of my impressions, speculations, and frustrations. The centrality of my role in the research study made self-reflection essential for monitoring the quality of the research design and ongoing analysis.

In an effort to establish the trustworthiness of the results, I additionally administered an end of seminar survey called the *Classroom Community Scale.* Moreover, I conducted individual interviews with participants during the first week of the spring 2009 semester. These data collections procedures are discussed next.

Classroom Community Scale

The survey instrument, *Classroom Community Scale,* was chosen because of its focus on elements characteristic of community in general and community specific to the classroom (Appendix A). Permission to use the instrument was obtained (Appendix B). Developed by Rovai (2002), the survey addresses essential traits of a community. They include: feelings of connectedness, cohesion, spirit, trust, and interdependence among members. Furthermore, the survey reflects the purposeful nature of community discussed in the literature review. Survey items account for "feelings regarding interaction among community members as they pursue the construction of understanding and the degree to which members share values and beliefs among each other regarding the extent to which their educational goals and expectations are being satisfied" (p. 201). To ensure content and construct validity, the development of the instrument also included a review by a panel of experts (three university professors who taught educational psychology) and a factor analysis. Additionally, estimates of reliability for the instrument were excellent (Cronbach's coefficient α for the full *Classroom Community Scale* was .93 and the equal-length split-half coefficient was .91).

Interviews

Hatch (2002) defines "formal interviews" as "planned events that take place away from the research scene for the explicit purpose of gathering information from an informant" (p. 94). Furthermore, formal interviews have the qualities of being: "structured" for the reasons that they occur at an established time and the interviewer assumes the role of leader; "semi-structured" in that the interviewer may deviate from the pre-set guiding questions to follow leads of interest; and "in-depth" or probing in nature. As is common in qualitative research, the interviews were "used alongside other data collection methods" with the goal of "explore[ing] more deeply participants' perspectives on actions observed by [the] researcher" (p. 91). In this case, the interviews were used alongside all of the data collected during the seminar (audio recordings, student work, surveys) to better understand participants' perspectives of their participation in the seminar, especially as it related to their *learning of mathematical proof* and *sense of belonging to a mathematical learning community.* Hatch calls the questions prepared in advance of the formal interview and designed to guide the conversation "guiding questions." For interviews conducted in conjunction with other data collection, as in this study, the researcher uses "ongoing analyses of observational data as a basis for constructing questions" (p. 101). As such, I created the guiding questions (Appendix C) after I completed transcription of all paired discourse samples. During the first week of the spring semester of 2009, I conducted formal interviews with seven of nine participants in the study (two declined participation).

Data Analysis

The second part of the methodology encompasses the analysis of the data sources. Ultimately, in qualitative research the process of data analysis unfolds in the mental work of the researcher. This means that interpretation—the researcher's best efforts to make sense of the data—is an inherent part of all qualitative work (Hatch, 2002). To aid in a close examination of the data, I used multiple analysis tools. These tools are source specific and allowed me to get close to the data. Upcoming sections describe the tools in detail. Table 3 also provides a summary of the data source and corresponding analysis tool.

Table 3

Alignment of Research Questions with Data Collection and Data Analysis Methods

Research Question	*Data Collection*	*Data Analysis*
#1 Focus on *learning*	Audio Recordings of Small-Group Discourse	Focal and Preoccupational Analysis
	Interviews	Typological Analysis
#2 Focus on *community*	Interviews	Typological Analysis
	Classroom Community Scale	Descriptive Statistics

As many qualitative researchers suggest, data analysis should begin, if only informally, as soon as data is being collected (Hatch, 2002; Merriam, 2002). Thus, before any formal analysis occurred, I engaged in memo writing as an initiatory type of analysis. Whereas the field notes are *descriptive records* of what occurred, memos are *records of analysis* written after leaving the field (Corbin & Strauss, 2008). The act of writing memos is an analytic session. It serves as a stimulus for gaining insights into the data. Memo writing forces the researcher to move the study from raw data to concept building. In the study under discussion, I wrote memos following each seminar session and during transcription sessions—all prior to the more formal analyses described next.

Focal Analysis and Preoccupational Analysis

The focal analysis and preoccupational analysis tools were recently developed by Sfard and Kieran in their work on cognition as communication (see Kieran, 2002; Sfard, 2002; Sfard, 2008; Sfard & Kieran, 2001). Taken together, these tools enable an investigator to examine the effectiveness of communication, a precondition of a learning interaction. In the case of the data source of audio recordings of paired discourse, the activity of communication is the unit of analysis. Focal analysis is a means for distilling the object-level "features of discourse that can count as indicators of its effectiveness or the lack thereof" (Sfard & Kiearn, 2001, p. 50). In focal analysis, a general comparison is made between what is said and what is done. This gives a detailed look at the mathematical content of the conversation. What is said is more formally known as the pronounced focus of the discourse. What is done, or the related activity, is considered the attended focus of the discourse. The pronounced and attended foci are public in nature. By comparing the pronounced and attended foci of the interlocutors, one can begin to look for places where effective communication breaks down. The third and most important focus of discourse for effective communication, the intended focus, is private. Thus, it is impossible to make comparisons of the intended foci of two interlocutors. We can at best make interpretations of intended foci. Nonetheless, this tripartite theoretical construct provides a means for beginning to analyze the effectiveness of mathematical discourse and the related interactive learning. Overall, focal analysis is a cognitively oriented type of analysis tool that helps the researcher to extract an understanding of students' mathematical learning from transcripts.

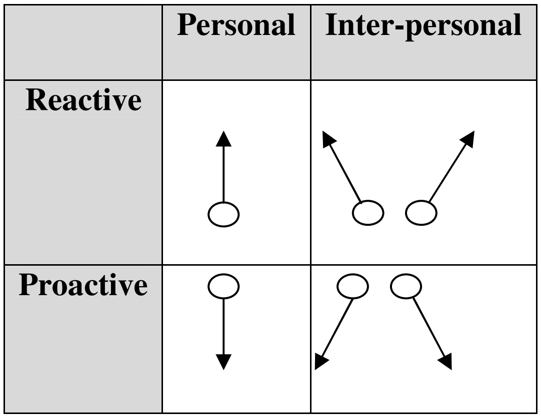
I used a template similar to the one in Figure 1 to conduct focal analysis. Audio-recordings were transcribed and recorded in the column labeled "pronounced focus." I referred to student work to describe the "attended focus." Taken together, the pronounced and attended foci informed possible interpretations of the "intended focus." This method is a slightly modified version of Sfard and Kieran's method, with the main difference being the use of audio recordings instead of video recordings.

Pronounced Focus	Attended Focus	Intended Focus
Transcript of audio recordings of classroom discourse	Student work	Interpretations by investigator

Figure 1. Template for focal analysis.

Preoccupational analysis is concerned with participants' engagement in conversation. It "deals with the question of how the participants of a conversation move between different channels of communication (private and interpersonal) and different levels (object-level and meta-level)" (Sfard & Kieran, 2001, p. 57). To examine the essence of real dialogue between students, I utilized the interactivity flowchart created by

Sfard and Kieran. In this analysis, utterances are classified as reactive or proactive. That is, we are interested in whether the interlocutor reacts to a prior comment or makes a response-inviting comment. Utterances can be further classified as personal or interpersonal. The interlocutor could react to a partner's previous comment or invite a response from the partner, thus contributing to interpersonal dialogue. On the other hand, the interlocutor could respond to his or her own previous comment or make a comment meant for his or her own ears, in effect talking with his or her own self. The classification scheme consists of using arrows emanating from small circles, which represent the utterances. The direction and slant of the arrow indicates the reactive/proactive and personal/interpersonal nature of the utterance. Object-level and meta-level utterances are indicated by whether the arrow is bold or dashed. A classification scheme for the arrows is found in Figure 2. To represent the conversation, a three-column interactivity flowchart is constructed. In the first two columns, the arrows representing each of the partners' utterances are represented singly. In the third column, the arrow patterns are combined to illustrate the overall dialogue. Sfard and Kieran (2001) view the combined use of the focal analysis and preoccupational analysis tools as a way to unify an analysis of the cognitive and social processes. The tensions between cognitive constructivism and social constructivism were discussed in Chapter 2. Sfard and Kieran believe that taken together the tools will provide insights that might have gone unnoticed with the use of more traditional approaches.



Object-level re- or pro-active utterance
Non-object-level re- or pro-active utterance

Figure 2. Interactivity flowchart for preoccupational analysis.

Typological Analysis

The interviews were analyzed using a model of analysis that Hatch (2002) refers to as *typological analysis.* “Typologies are generated from theory, common sense, and/or research objectives, and initial data processing happens within those typological groupings” (p.152). The choice of typologies, according to Hatch should be obvious when typological analysis is the appropriate analysis strategy. “Typological analysis only has utility when initial groupings of data and beginning categories for analysis are easy to identify and justify” (p. 152). The initial typologies in this study stemmed from the research questions. Specifically, they centered on the relationship of the seminar experience to participants’ *learning of mathematical proof* and to their *sense of mathematical community*. The following steps, offered by Hatch, were used to guide the analysis of the interview data.

1. Identify typologies to be analyzed.
2. Read the data, marking entries related to your typologies.

3. Read entries by typology, recording the main ideas in entries on a summary sheet.
4. Look for patterns, relationships, and themes within typologies.
5. Read data, coding entries according to patterns identified and keeping a record of what entries go with which elements of your patterns.
6. Decide if your patterns are supported by the data, and search the data for nonexamples of your patterns.
7. Look for relationships among the patterns identified.
8. Write your patterns as one-sentence generalizations.
9. Select data excerpts that support your generalizations. (p. 153)

Survey Analysis

The *Classroom Community Scale* contains 20 five-point Likert scale items. Raw scores range from 0 to 80. Higher scores are interpreted as a stronger sense of classroom community. Subscales ranging from 0 to 40 also exist for connectedness and learning. The investigator calculated the overall raw score and the subscale raw scores of each participant. The findings are reported using descriptive statistics.

Trustworthiness

In applied fields, such as mathematics education, teachers, administrators, and policymakers look to research findings to improve practice. This places an impetus on the researcher to demonstrate that the study was "conducted in a rigorous, systematic, and ethical manner such that the results can be trusted" (Merriam, 2002, p. 24). While research traditions grounded in a quantitative approach have well established measures for ensuring validity and reliability, there exists ongoing debate in the qualitative community as to criteria for addressing the same (Merriam). Nonetheless, the researcher has the responsibility of explicating the efforts taken to enhance the study's validity and reliability and to discuss any ethical considerations of the study.

The epistemological underpinnings of qualitative research are such that internal validity is an inherent strong point. Internal validity deals with how closely the findings resemble reality. Whereas positivist researchers attempt to find reality "out there," qualitative researchers ascribe multiplicity and unique individual constructions to the concept of reality. Rather than employing an instrument designed to capture reality, qualitative researchers attempt to *understand* reality through direct observation. They do this by immersing themselves in the natural setting and by interviewing the setting's participants. Thus, in its alignment with an interpretive understanding of reality, the qualitative approach to inquiry becomes a natural first step in establishing a study's internal validity (Merriam, 2002).

The issue of reliability is one of replication. Would a researcher repeating the study produce the same findings? However, when the purpose of a study is to understand the meanings which humans bring to a particular context—meanings that are mutable—reliability in the traditional sense becomes a more or less moot point. Qualitative researchers instead conceive the issue in terms of "dependability" or "consistency" (Lincoln and Guba, 1985, p. 298-299). In other words, they are interested in *"whether the results are consistent with the data collected"* (Merriam, 2002, p. 27). Qualitative research is not judged then on its reproducibility, but rather on whether the results make sense in relation to the data.

In interpretive qualitative work, such as the study under discussion, triangulation is a commonly recognized strategy for ensuring both validity and reliability (Merriam, 2002). It is the means by which the researcher confirms or cross-validates the emerging findings. To ensure validity and reliability, this study utilized the triangulation strategy of multiple methods, one in which "the researcher collects data through a combination of interviews, observations, and document analysis" (p. 25). Within the commognitive analysis portion of the study, the investigator cross-referenced audio recordings of paired discourse with students' written work. The use of multiple methods in this study contributes both to the consistency/dependability of its results and to a portrayal of the phenomena under study that is closely aligned with the realities of the participants.

Due to a notion of generalizability that stems from a well-established positivist approach to research, qualitative researchers often encounter challenges in justifying the external validity of their results. Because qualitative researchers work with small non-random purposefully selected samples, they are unable to make inferences about populations comparable to quantitative researchers who work with random samples (Merriam, 2002). Qualitative and quantitative approaches to inquiry, however, stem from different research purposes. The goal of a qualitative approach, and hence the small sample, is deep understanding over large-scale prediction. Consequently, Merriam frames generalizability in qualitative researcher in the following way: "The general lies in the particular; what we learn in a particular situation we can transfer to situations subsequently encountered" (p. 28). The responsibility of generalizing lies with the reader—"readers themselves determine the extent to which findings from a study can be applied to their context" (p. 29). Patton (1990) refers to this notion as "context-bound extrapolations rather than generalization" (p. 491). It is incumbent upon the researcher, then, to provide the reader with *thick, rich description* so that they can make informed comparisons between their own context and the one reported. The current reader is encouraged to determine the extent to which the findings from this study can be applied in their specific situation. To that end, the investigator has endeavored, at every corner, to provide an adequate database coupled with ample description. In the findings, the reader should find the investigator's interpretations fully supported by the data. Moreover, the reader will find the transcripts of the paired-discourse under analysis in the appendices of the manuscript.

Ethical implications are the final, and perhaps overriding, consideration. In particular, issues related to informed consent, privacy, and protection from harm are significant. To ensure the protection of participants against any ethical wrongdoing the following measures were taken: 1) the investigator gained permission to conduct the study from the Institutional Review Boards of Indiana University of Pennsylvania and the university where the study took place; 2) the investigator provided potential participants with a written and verbal description of requirements of participation in the study; 3) the investigator collected written consent forms from all participants and; 4) participants were free to withdraw from the study without penalty.

Summary

In summary, this study employs multiple layers of data collection and data analysis in an effort to gain insights into the nature of mathematical discourse as it relates to participants' learning of mathematical proof and their sense of mathematical learning community. The theoretical framework of the study grounds the methodology. That is, the methodology accounts for the symbiotic relationship between mathematical learning, defined as an initiation to the special form of communication that is mathematical, and the broader learning environment. A defining feature of the methodology is the integration of focal and preoccupational analysis, two new and sophisticated methods for analyzing mathematical discourse. This integration allows for a microscopic investigation into mathematical discourse and learning of mathematical proof, while simultaneously maintaining a wide-angle view of the related social context.

CHAPTER 4

FINDINGS

The aim of this study is to begin to uncover the nature of undergraduate mathematics majors' learning of mathematical proof by examining their mathematical discourse. In addition, the study seeks to describe students' sense of learning community in relation to their participation in a seminar on mathematical proof utilizing small-group discourse. The study drew on multiple sources of data. This chapter presents the results of the analysis of the data. The first section gives an individual and detailed analysis for five small-group discourse excerpts from the seminar. The second section is comprised of a summary of the interview data. The third section reports the results from the *Classroom Community Scale* survey administered to the participants.

Small-Group Discourse Analysis

Background

The freshman-sophomore mathematics seminar met ten times during the fall semester of 2008. The freshmen and sophomores were also encouraged to attend the senior mathematics seminar presentations during the final week of the semester. Topics of the initial four meetings included an introduction to the field of mathematics, a presentation by the career services office, and a brief introduction to logic. The latter six meetings centered on mathematical proof using three chapters from Daniel Solow's *How to Read and do Proof* as a basis for discussion. Typically, the seminar opened with a brief whole-group discussion on the week's reading or the previous week's work. Then students were assigned to work in small groups, usually consisting of two or sometimes three students. The group assignments changed each week.

Over the course of the six weeks, I collected and subsequently transcribed seventeen digital audio-recorded samples of the participants' small-group discourse. Hatch (2002) maintains that inductive information processing is characteristic of all qualitative research and that "all inductive analysis must begin with a solid sense of what is included in the data set" (p. 162). Accordingly, I next read and re-read the collection of transcripts in its entirety, as well as related memos, to get a sense of the whole and recorded my impressions. From this initial analysis, I chose five excerpts to undergo further analyses using the tools of Sfard and Kieran (2001). My criterion for choosing the five excerpts was two-fold. First, each sample needed to exhibit fullness. That is, the discourse needed to maturate over time, but not necessarily draw to a close. Practicalities such as limited time, the particulars of an assigned task, and different student rates of working detracted from the fullness of some samples. We see, however, evidence of evolution in each of the excerpts chosen. Second, and perhaps most importantly, in each of the excerpts, I identified specific features that aroused my attention during the initial analysis. Thus, much like the purposeful selection of the "case" in case study research, each sample "exhibits characteristics of interest to the researcher" (Merriam, 2002, p. 179). In summary, I chose the excerpts with depth in mind—believing, like Sfard and Kieran (2001), "that a close and detailed picture of one little sample may sometimes be more revealing than a lengthy study with hundreds of participants. When using a microscope one may discover a whole new world of complex relationships and rich phenomena" (p. 70).

Those excerpts meeting the selection criterion then underwent further analysis using the complementary tools of focal and preoccupational analysis. Focal analysis, "gives us a detailed picture of the students' conversation on the level of its immediate mathematical contents and makes it possible to asses the effectiveness of communication," while preoccupational analysis allows for the examination of "participants' engagement in the conversation" and hence clues about its effectiveness (Sfard & Kieran, 2001, p. 42). I began the work of each focal analysis equipped with the template found in Figure 1 in Chapter 3, the transcript, the audio recording of the excerpt, and the accompanying student work. To discover participants' attended focus and make consequent interpretations, I listened to the audio recording many times over as I concurrently read the transcript and scrutinized the participants' work. The importance of listening to the audio recording during analysis cannot be underestimated. This revealed critical features of the discourse that would be missed in reading the transcript alone, such as the interlocutors' intonations and pauses in conversation. Similar to its value in focal analysis, the audio recording was a vital piece, (in addition to the transcript and template in Figure 2) in completing the preoccupational analysis for each excerpt.

The interpretive judgments of a researcher imbue all levels of qualitative research (Hatch, 2002). They are, in the words of Hatch, "the researcher's best efforts to produce meaning that makes sense of the social phenomena they are studying" (p. 180). Sfard and Kieran (2001) specifically address the interpretive status of claims made using the tools of focal and preoccupational analysis. A review of their three-part theoretical construct for discursive focus is in order. First, the *pronounced* focal element is public (utterances). The *intended* focal element is private. It includes the "cluster of experiences evoked by other focal components plus all the statements a person would be able to make on the entity in question" (p. 53). Finally, the attended focal element is made up of "the image a person perceives (or imagines)" and also the "attending procedure she is performing while scanning this image" (p. 53). The attending focal element is the mediator between the private and public elements. The private nature of intended foci makes them impossible to compare. Thus, any judgments that I make about the effectiveness of communication, which largely depends on interlocutors' intended foci, are purely interpretive. The judgments, based on the intended foci brought to *my* mind upon hearing the interlocutors' utterances, "should always be regarded not more than the best hypothesis" that I was able to produce (p. 53).

On a whole, the interpretations made throughout this chapter relate to the *effectiveness* of communication (Sfard, 2008) and its *productivity* (Sfard & Kieran, 2001). If communication is taken as a rule-regulated and patterned activity, then effective communication is the successful interplay of interlocutors' actions and re-actions that falls within these pre-defined boundaries. Those doing the communicating and those looking on may very well judge the success of the interplay differently—thus the interpretive aspect of effectiveness. Confidence in the effectiveness of communication, however, is actually a pre-condition for participation in communication. In layperson's terms, interlocutors need to feel as though they are "on the same page." We assume that communication is proceeding effectively until we have reason to believe otherwise. Thus, the task of the researcher is not to try to demonstrate that communication is effective, but rather identify where it breaks down. The effectiveness of communication no doubt bears on its educational productivity. Sfard and Kieran (2001) state: "In the case of mathematical discourse, an interaction will be regarded as educationally productive if it is likely to have a *durable* and *desirable* impact on students' future participation in this kind of discourse" (p. 50). We cannot see into the future though. Thus, the researcher must look at, for example, how discourse has led to a solution of a problem or if it has become richer in its use of rules and concepts, to make judgments about its overall lasting effect.

Following the focal and preoccupational analyses, I began the writing process. Hatch (2002) argues that writing is another stage of data analysis. He writes:

> Writing involves a special kind of thinking that is hard to do except during the act of constructing meaning in text. As writing proceeds, you will likely see relationships, patterns, and themes in new or different ways. When this happens, it will mean a return to the data to be sure that what's new or different is supported there. You should expect that findings will be shaped by the writing process. (p. 225)

Similar to Hatch's explanation, there existed reciprocity between my writing and analysis. The focal and preoccupational analyses formed the basis for beginning to write. However, the writing process served to intensify the initial analyses. Hatch also argues that the wide variety of analytic models in qualitative research results in an array of forms for writing the research findings. Here, comparable to case study research, I fashion the findings as comprehensive descriptions of each excerpt. In so doing, the generalizability of the findings diminishes. Merriam (1988), however, points out that there is much to learn from particular cases. It is in this spirit that the findings from analyzing the small-group discourse are now presented. The excerpts make use of pseudonyms to maintain the confidentiality of participants. For each of the excerpt's narratives, the reader is strongly encouraged to read the accompanying transcript provided in the appendices. However, every effort has been made to provide the reader with pertinent details of each excerpt's narrative. Moreover, brackets are used in the narrative to refer the reader to the line(s) in the transcript under discussion. Finally, readers less familiar with mathematical proof are invited to review Appendix D for a summary of the key terms and strategies.

Excerpt A

Overview

A discussion on Excerpt A is a fitting place to begin. Not only is it chronologically appropriate—it is a sample discourse of the very first proof that students attempted during the semester in pairs—it also points to the potential value of discourse analysis for researching the complexities of learning of mathematical proof, particularly in a collaborative setting. Like most research endeavors, the analysis of the excerpt raises more questions than it answers. Nevertheless, the discourse, as the unit of analysis, may allow us to raise questions (and hypotheses about their answers) previously inaccessible through other methods of analysis.

Excerpt A is from the first seminar on proof. Before coming to the seminar session, students were asked to read the second chapter of the Solow text on the forward-backward method. The conversation, a little under 14 minutes in length, occurs between Jacob, a freshman male, and Sherri, a sophomore female, randomly assigned to work together. Neither student had experience in a proof-based college mathematics course. In the excerpt, the students are working on Problem 2.23 from Solow's text (see Figure 3). The reader is invited at this time to read the transcript found in Appendix E and examine the work in Figure 4 and Figure 5 to aid in understanding the analysis that follows.

Prove that if the right triangle XYZ in figure is isosceles, then the area of the triangle is $\frac{z^2}{4}$.

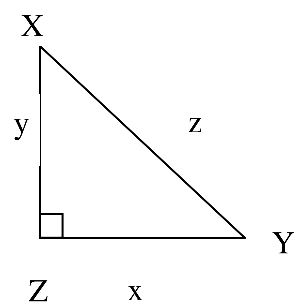


Figure 3. Problem 2.23 from Daniel Solow's *How to Read and Do Proofs* (4th ed.).

Prove that if the right triangle *XYZ* in Figure 2.1 is isosceles, then the area of the triangle is $\frac{z^2}{4}$.

A: The right Δ XYZ is isosceles

B: The area of the Δ is $\frac{z^2}{4}$

Key Question: How do you show the area of a triangle.

Abstract Answers:

A1: $x^2 + y^2 = z^2$	Pythag therom
A2: $x = y$	def of isoceles Δ
A3: $x^2 + x^2 = z^2$	substitution
A4: $2x^2 = z^2$	substitution
A5: $x^2 = \frac{z^2}{2}$	simplifying
A6: $x = \sqrt{\frac{z^2}{2}}$	solve for X
A7: $a = \frac{1}{2} x y$	area of a Δ
A8: $a = \frac{1}{2} x^2$	
A9: $a = \frac{1}{2} \left(\sqrt{\frac{z^2}{2}}\right)^2$	
A10: $a = \frac{1}{2} \left(\frac{z^2}{2}\right)$	
B1 $a = \frac{z^2}{2^2}$	
B $a = \frac{z^2}{4}$	

Figure 4. Jacob's work on Problem 2.23.

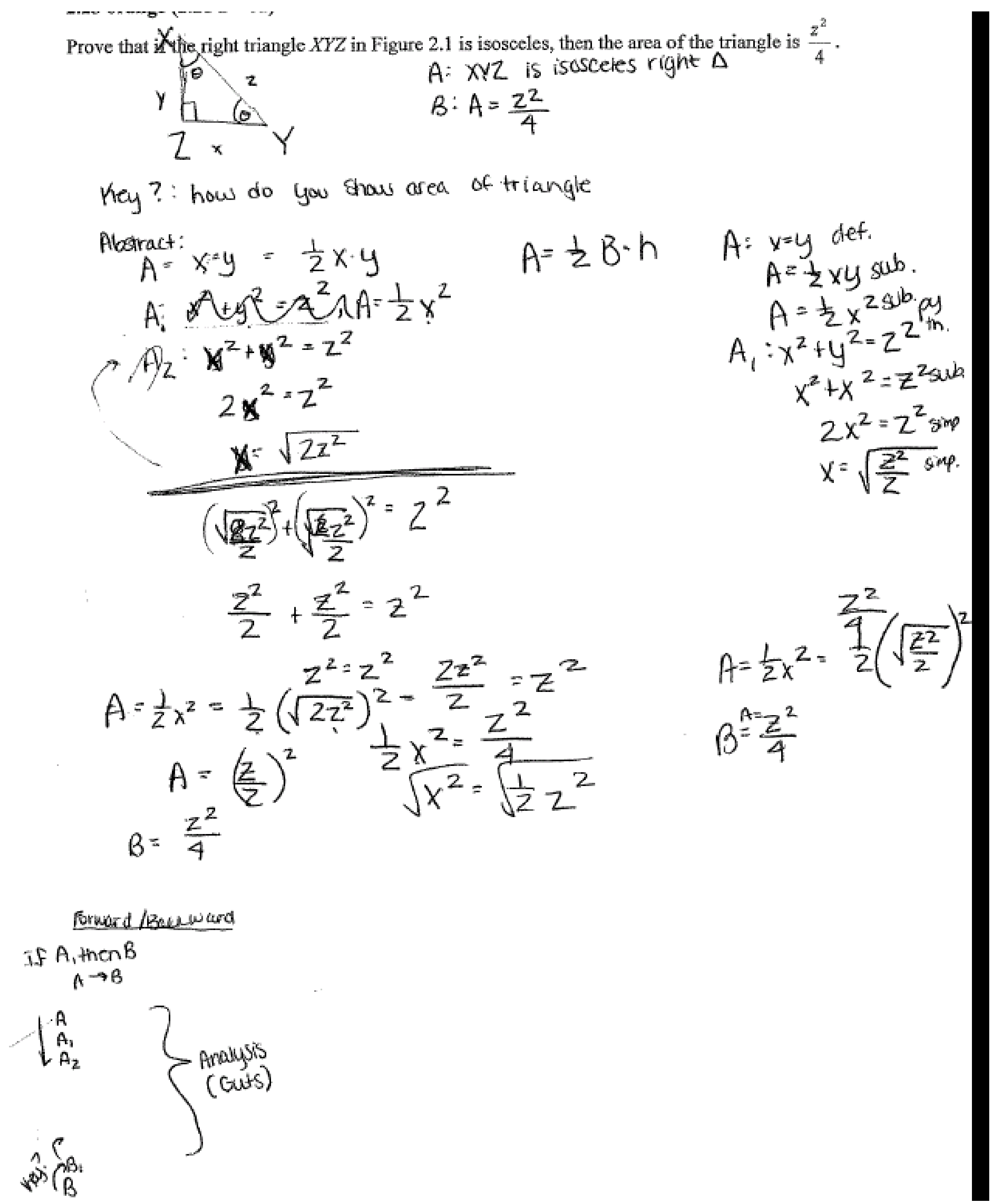


Figure 5. Sherri's work on Problem 2.23.

Unlike the findings presented for forthcoming excerpts, we begin the discussion of findings for Excerpt A at the discourse's end, rather than at its beginning. In the time allotted, Sherri and Jacob produce a proof analysis (set of steps) that convincingly demonstrates the conclusion of the proposition when the hypothesis is assumed true. The pair did not complete a condensed proof and even indicated some resistance to attempting one [85]. However, other pairings did not complete the analysis, let alone a condensed proof. Thus, if the measure of the group's accomplishment is based on the end-product — the series of logical steps and the justifications that they produced— it was a success. Naturally, we are interested in the elements of discourse that lead students to the successful completion of a proof task. Let us now sift through the residue of Jacob and Sherri's discourse with the tools of focal and preoccupational analysis to see what we might learn.

The transcript begins with Sherri and Jacob discussing key questions [1-7], a strategy brand new to them. Their utterances can be classified both as non-object level [1, 3, 4a, 5a] and object-level [2, 4a, 5b, 6]. Comments such as "are you serious"[3] and "that's really dumb and really obvious"[4] do not advance the mathematics of the conversation. Instead, they are laden with social implications; serving as conversation fillers and perhaps as relationship "test balloons." The comments may merely be the students' value judgment on the usefulness of key questions. However, it seems likely that, working together for the first time early in the semester, Sherri and Jacob are additionally assessing each other's level of commitment/value of working on proof in the seminar format, so that they know how to negotiate their work together.

Jacob sets the mathematical dimension of the discourse in forward motion, quickly posing two key questions [2] before settling on a third "How do you show the area of a triangle?"[4]. Both students write it on their papers, Sherri while saying it aloud [5a]. While their wording of the question indicates an intended focus on the conclusion, it is potentially limited in how it will move the proof process forward. We know that the area of a triangle is one-half of the product of its base and height. To complete the proof successfully, the students need to demonstrate that the area is equal to a *specific value.* Thus, in terms of mathematical precision, a better-crafted key question would have been: How do we show the area of a triangle is equal to a certain number? Moving on, Sherri utters, "abstract answer" [5b], and Jacob's response [6] suggests that he is attempting to clarify his understanding of this newly introduced concept. However, it becomes patent to Jacob [8] that Sherri is bypassing the abstract answer to begin working with the given (hypothesis). Neither student records an abstract answer on their paper, nor does the abstract answer resurface in their remaining discourse. An appropriate abstract answer, incidentally, would have been to show one-half of the product of the triangle's base and height equal to a number, namely $z^2/4$.

It is worth pausing to consider the first eight lines of transcript, just described, in the broader context. With a limited key question—never referred back to in the discourse—and no abstract answer, the students generate a feasible proof analysis. Perhaps the familiar mathematical content (right isosceles triangle) of the proposition failed to motivate more attention to the key question and abstract answer on the part of the students. After all, Sherri seems very anxious to, as Jacob says, "Jump right in" and work with what she knows. And, it does not take long, after outlining what she knows [9, 17, 19], for her to state, “Oh, this one’s really easy” [21]. Earlier in the seminar session, Sherri responded to the question “In your own words, what is mathematical proof?” She wrote: “A proof is a routine or order of steps to go through to prove a theorem is true.” Thus, much like Moore’s (1994) observations about undergraduates’cognitive difficulties with proof**,** Sherri views the task more like a problem to be solved, rather than the development of a mathematically convincing argument. It is not surprising then that, after arriving at the final step of the analysis, showing that the area of the triangle is $z^2/4$, she claims to Jacob, “I got it. I’m not writing it out though in paragraph form” [85].

It could be argued that Sherri’s trial-and-error method had the semblance of authentic mathematics. Rather than adhere rigidly to a prescribed strategy (key question, abstract answer, etc.), she plays, in the mathematical sense, with what she knows. And when she does not come upon what she is looking for, she is willing to revise and play some more. Lines 34-40 of the transcript are a delightful illustration of this approach. Knowing that the right triangle she was working with was isosceles (see Figure 5), Sherri substitutes *y* for *x* in the Pythagorean Theorem and then solves for y^2. Substituting the resulting expression, $z^2/2$, back in to the Pythagorean Theorem yields $z^2 = z^2$ and much laughter. After admitting to “going in a big circle” [40], Sherri takes up “playing” with a new formula, the area formula. Sherri’s playing and perseverance, along with Jacob’s guidance [e.g., Line 79] eventually lead her to showing the area of the triangle equals $z^2/4$, that which was to be demonstrated.

That Which was to be Demonstrated, but What About That Which <u>Could</u> Have Been Learned?

Sherri arrives at a set of steps constituting a coherent proof analysis. But did Sherri *learn* anything about mathematical proof from this experience? In other words, did the discourse move Sherri toward a more sophisticated participation in discourse on proof? Was the discussion educationally productive? Certainly, as a sophomore mathematics education major, the algebraic manipulations/substitutions required of the proof were not new to her. Perhaps, for Sherri, the particular proof was a poor instructional choice, as the algebraic familiarity only seemed to reinforce her posture toward proof as a routine or order of steps to go through. On the other hand, a close analysis of the discourse suggests there existed significant opportunities, herein named "discursive entry points," that had they been handled differently by the interlocutors, may have led to greater learning for Sherri.

Sherri gets quite a chuckle when she comes across the statement of reflexivity ($z^2 = z^2$) in her work. "Apparently that wasn't right" [35], she laughs. There is no indication from her utterances, however, that she takes the time to determine *why* she got the result she did. She is looking for the *right* answer and it is *apparent* that she did not obtain it. Thus she moves on to try something else. Revisiting/revising the key question and its answer at this natural break in the discourse may have been particularly powerful for Sherri on two levels.

Sherri's utterances in relation to her strategic intentions. We can read between the transcript lines to determine that Sherri knows she must produce $z^2/4$. However, her discourse suggests that her methods are more haphazard than strategic. While Sherri may have down-played the usefulness of posing a key question and abstract answer when she began, the ground is now fertile for revisiting their *value* in guiding the work. This re-visit could have been initiated by someone (Sherri or another interlocutor) asking the question "why" immediately following Line 35 in the transcript. Sherri's answer was "not right" because she substituted a value back into the same equation she used to solve for it. But she might have avoided this altogether had she referred to the question "How do I show the *area* of a triangle (is equal to $z^2/4$)?" and its answer "Show that half the product of the base and height is equal to $z^2/4$," to steer her initial substitution. The particular entry point into the discourse [after line 35] discussed above, presents a potential opportunity to help Sherri move beyond a trial-and -error mentality, which might not be as effective an approach when the moves are not algebraically familiar. This by no means diminishes the value of trial and error in the proof process, but rather uses it as a springboard toward increasing mathematical sophistication.

Moving Sherri to a more holistic discourse on proof. Moving Sherri from a discourse on proof more or less discrete in nature ("routine or order of steps to go through") to a more holistic discourse on proof (proof as an endorsed narrative or objective truth) is no doubt a long term endeavor. It is certainly worth exploring what, if any, aspects of Excerpt A lend themselves to this end. The discourse entry point just discussed (immediately following Line 35), may be one opportunity of many needed to remind Sherri of the "big picture." Would continual reminders to focus on the key question in this discourse and subsequent discourses foster a more expert understanding of proof for Sherri? Would it help her to view the order of steps as a cohesive whole? Alternatively, would the emphasis only serve to reinforce a narrow focus on the final step? These questions are exemplary of the questions that studying the learning of mathematical proof through discourse raise and have the potential to answer.

What about Jacob?

The discussion of the findings to this point has mainly focused on Sherri and on a particular break in the discourse that might have served as a window of opportunity to nudge her to increased mathematical sophistication. But, what about Jacob? How did the two interact and contribute to each other's more expert participation of mathematical proof?

The reader of the excerpt may get the sense that Jacob, unlike Sherri, was more open to employing some of the strategies introduced at the beginning of the seminar session. It is true that he refers to the key question(s) as "really dumb and really obvious"[4a]. However, there are multiple utterances throughout the discourse [2, 4b, 6, 8, 41], that indicate Jacob's attended focus on process, suggesting that his original comment[4] may have been more socially motivated than representative of his willingness to try the strategies. In fact, Jacob seems slightly offset by the fact that Sherri moves so quickly to working with the given. "Alright," he says, "we're just going to (solve/simplify?) Jump right in?"[8]. After mainly attending to Sherri's utterances [9-33], speaking up at times to offer alternative algebraic representations [24, 27] or checking an algebra error [29], Jacob announces that he is going to start working backwards [41]. Ultimately, the decision leads him to almost immediate gratification (see Figure 6). Through algebraic manipulation, he quickly links his conclusion B (a = $z^2/4$) to his last forward step ($a= \frac{1}{2} x^2$) (see Line A8-B on Jacob's work, Figure 4). He seems genuinely pleased with the results of his decision to work backwards exclaiming, "It's wild check that out" [47]. Sherri is not a participant in Jacob's maiden venture through the backwards method. In fact, during his work and despite invitations for her to join him [43, 45], Sherri pursues her own line of thinking. An extract of the preoccupational analysis shown in Figure 7 highlights the uni-dimensional nature of each of their utterances during what is a mathematically crucial point in their conversation. The several vertical arrows suggest that, much like toddlers parallel play, these students are working side by side but ultimately pursuing their own ideas.

41	Jacob	I'm going to have to say. . . I'm going to start working backward
42	Sherri	Wait, area equals. . . one-half base times height.
43	Jacob	See, Check it out, here's what I'm going to do.
44	Sherri	The base and the height are the same. Yea so then (Sherri is talking to herself while Jacob is talking)
45	Jacob	Yea, here's what I'm doing.
46	Sherri	have x times
47	Jacob	z squared over 2 squared. . see check that, and we can even go a step further,(Sherri says "wait"), a= z over two, the squares out there. How'd you (like?) that. It's wild check that out
48	Sherri	a)one-half x squared, should be area, one half x squared (this is said under her breath to herself)(then louder she says) the area equals one half x squared. . b)Wait what are you doing?

Figure 6. Lines 41-48 from Excerpt A.

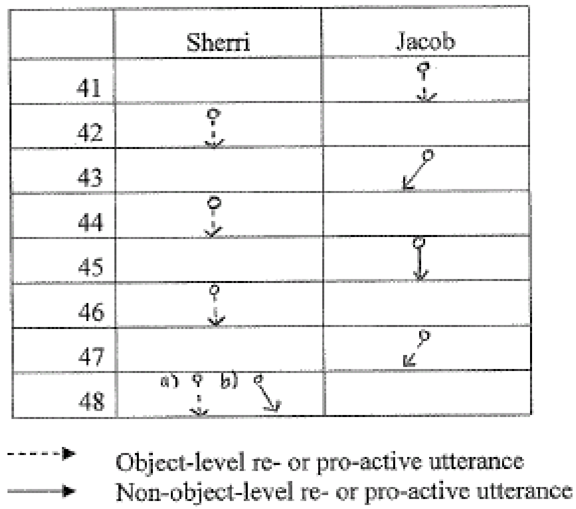


Figure 7. Preoccupational analysis of lines 41-48 of Excerpt A.

Sherri eventually expresses interest in what Jacob is doing [49b]. However, it may be too late since she misses the “aha moment.” She has access to the discrete steps that she was looking for, but did not *participate* in Jacob’s backward approach. To what extent did participation in the discourse benefit each of the interlocutors? What, in other words, was the nature of their collaboration?

The discourse provides evidence that the interlocutors may each have profited from the other’s assistance in practical matters and in content areas in which they had prior knowledge. For instance, early in the discourse, Jacob asks, “How do you know z’s the hypotenuse? Is that how it always is?”[10]. Jacob, no doubt, knows that the choice of a variable to label a figure is arbitrary. It seems that he was about to use the variables *a, b,* and *c* to represent the sides of the triangle [9]. Sherri reminds Jacob, albeit with slight exasperation, to consult the given figure [11]. It would have been difficult for Jacob to show the area of the triangle equal to $z^2/4$ while working with a hypotenuse of length *c*.

Later in the discourse, Jacob quickly notices Sherri's error in solving for *y* [28, 29], something Sherri is quite capable of doing [30]. This is a significant catch on Jacob's part, as substitution of the wrong value will inevitably not lead to what is sought. We, in fact, see this later in the discourse when Sherri mistakenly substitutes the original value that she found for *y* (also the value for *x* because the triangle is isosceles) into the area equation [72]. Through dialogue [73-78], Jacob again helps her to correct her mistake [79]. It is likely that, in time, each of the interlocutors would have resolved the aforementioned practical matters on their own. A partner's assistance, however, made it possible to address the matter efficiently, rather than having it become a distraction in the overall task.

Despite advantages related to efficiency in managing content, there is little to suggest that in this exchange either of the interlocutors increased the sophistication of their discourse as a result of their collaboration. Jacob likely derived an initial understanding of and willingness to employ the forward-backward method from his reading of Chapter 2 in the Solow text and in the whole-group discussion that preceded his pair-work with Sherri. Nothing in the discourse hints at Sherri's utterances lending Jacob any further insights in this area. Two junctures in the discourse, though, seem significant in terms of Jacob's potential influence on Sherri's thinking about managing the proof process. The first comes in the form of two innocuous questions [8] about Sherri's intention to move on rather than come up with an abstract answer. The second takes the form of an informal invitation [41, 43] to "check out" what he's doing. In both cases, however, Sherri appears to be absorbed in her own work, more or less oblivious to

Jacob's utterances. Consequently, neither juncture manifested into a shared discourse on the proof process.

Summary

Recall that Sfard (2008) defines a routine as "a set of metarules defining a discursive pattern that repeats itself in certain types of situations" (p. 301). The set is further divided into three subsets: applicability conditions, procedures, and closure. To be a skillful participant in mathematical discourse, one must have command over its metarules. A lasting change in one's use of metarules is a sign of mathematical learning. It is doubtful that the conversation in Excerpt A was educationally productive for Sherri in terms of producing a change in how she discusses mathematical proof. Her routine approximated one of trial-and-error in problem solving. She did not seem to recognize Problem 2.23 as a circumstance (applicability condition) different from problem solving. Her procedures and a lack of closure (she was dismissive of the need to try writing a condensed proof) followed suit. Jacob, on the other hand, seemed to recognize in Problem 2.23 (circumstance) an opportunity to try a new routine, namely the forward-backward method. His success may contribute, if even in a small way, to Jacob's learning of mathematical proof—to an eventual and lasting change in his mathematical discourse.

The analysis of this excerpt has brought to light complexities of learning mathematical proof in a social setting that student work alone may not reveal. While the proof analysis produced by the two students was successful in its own right, the analysis of discourse raises questions about how successful the paired-experience was in terms of advancing the students' thinking on mathematical proof. Several potentially influencing factors were raised. For example, it is probable that, working together for the first time, the students were partially using the discourse to manage perceptions about themselves. Moreover, the familiarity of *content* in the proposition may have failed to spur an interdependent focus on managing the proof process. Thus, nothing became of what were perhaps very fruitful opportunities to push the interlocutors' learning forward. In the next excerpt, we examine a paired-discourse that, unlike this one, falls short in producing a cohesive proof analysis.

Excerpt B

Overview

Excerpt B is taken from the second seminar session on proof. Thus, the forward-backward method was introduced the week prior and students were asked to read about using definitions and mathematical terminology in proofs (Chapter 3 of Solow) in preparation for the second session. The transcript of Sara and Lisa's work on Problem 3.9 (see Figure 8) is a little over 18 minutes in length. A whole group discussion on Problem 3.11 (see also Figure 8), a structurally similar problem, preceded the paired discussion. At this point, the reader is invited to read the transcript in Appendix F and study the accompanying student work in Figure 9 and Figure 10 in order to make sense of the discussion that follows.

Problem 3.9
Prove that if n is an odd integer, then n^2 is an odd integer.

Problem 3.11
Use Proposition 2 on page 25 to prove that if a and b are even integers, then $(a+b)^2$ is an even integer. (**Proposition 2**: If n is an even integer, then n^2 is an even integer.)

Figure 8. Problems 3.9 and 3.11 from Daniel Solow's *How to Read and Do Proofs* (4th ed.).

Key Question: How do I show a # is odd
Abstract Ans: $n = 2k+1$

A: n is odd integer

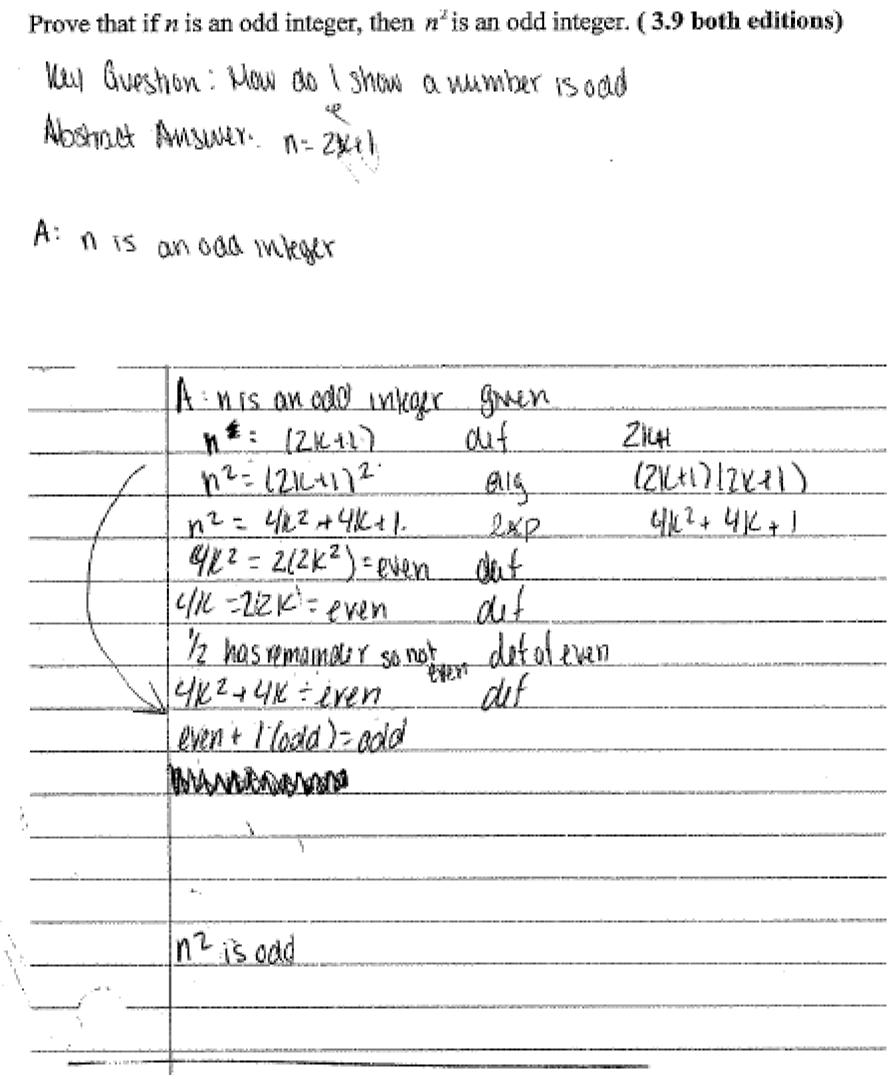
B: $n = 2k+1$

B: n^2 is odd

n is an odd integer	given
$n = 2k+1$	def of odd #
$n^2 = (2k+1)^2$	algebra
$n^2 = 4k^2 + 4k + 1$	algebra
$4k^2 = 2(2k^2)$	def of even #
$4k = 2(2k)$	def of even #
$4k^2 + 4k = 2(2k^2 + 2k)$	even ans

prove adding 1 makes it odd
↓
n^2 is odd

Figure 9. Sara's work for Problem 3.11(handout (top) and notebook paper (bottom)).



Prove that if n is an odd integer, then n^2 is an odd integer. **(3.9 both editions)**

Key Question: How do I show a number is odd

Abstract Answer: $n = 2k+1$

A: n is an odd integer

A: n is an odd integer given

$n = (2k+1)$ def $2k+1$

$n^2 = (2k+1)^2$ alg $(2k+1)(2k+1)$

$n^2 = 4k^2+4k+1$ exp $4k^2+4k+1$

$4k^2 = 2(2k^2) =$ even def

$4k = 2(2k) =$ even def

$1/2$ has remainder so not even def of even

$4k^2+4k =$ even def

even + 1 (odd) = odd

n^2 is odd

Figure 10. Lisa's work for Problem 3.11. (handout (top) and notebook paper (bottom)).

An overall examination of the transcript reveals two natural breaks in the paired discussion, leading to three phases of discussion. Lines 1-16 of the transcript might be called the planning phase. In this phase, we see evidence of the women attempting to apply the forward-backward method as they discuss their key question [2-5], abstract/applied answer [6-7,10-11], hypothesis (A)/conclusion (B) [9,11] and general plan of action [12-14]. Line 17 represents a distinct transition in the discussion from planning to active engagement in fleshing out the proof. This phase constitutes the majority of the overall discussion as the pair grapples to show that n^2 is an odd integer. Shortly after Sara states that their work is “going in circles” [107], Lisa suggests that they revisit their use of the forward-backward method [119]. This marks the beginning of the third and last phase of the discussion [119-131], in which they discuss the possibility of revising their strategy. I circulated to Sara and Lisa’s group just as time was up for the week’s seminar session. After assessing their progress and assuring them that their thought process was on the right track, I excused myself to pull the whole group together and end the seminar for the day. I returned to the women after they turned the audio recorder off and the other students were leaving. I discussed with them how they could rename the $2k^2 + 2k$ (found in Line 7 on Sara’s notebook paper and Lines 5-6 of Lisa’s notebook paper) as a new integer, say *l*, to arrive at $n^2 = 2l + 1$, thus demonstrating that which was to be shown, that n^2 was odd. More will be said in the next chapter about the role of the entrance of the teacher in students’ paired-discourse on proof. The forthcoming analysis, however, focuses on the pair’s discussion prior to my entrance into their discussion.

Phase 1: *Planning*

A close examination of what the interlocutors say and do allows us to comment on the effectiveness of their communication. Moreover, what is *not said* and *not done* by the interlocutors seems to jump to the foreground under the intense scrutiny of what *is said* and *is done.* It is within these protruding voids that we might learn the most from the data. At first glance, the planning phase [1-16] of Sara and Lisa’s discussion appears to be on target in terms of the aims of the first two seminar sessions. The women obviously attempt to utilize the strategies recently introduced to them in the seminar (developing, for example, a key question and abstract answer). What is more, the transcript suggests that, from the perspective of the women, the communication is proceeding effectively (note phrases of agreement in Lines 3, 5, 7, 11, and 16 and similarities of what is recorded on their handouts).

Phase 2: Engagement

It is interesting to note a distinct change in the discussion as it moves from the planning phase into the engagement phase [beginning with Line 17]. It seems as though maintaining effective communication requires more work on the part of the interlocutors. In other words, unlike the exchanges in the planning phase where utterances that moved the mathematics in the conversation forward were merely agreed upon [e.g., Lines 4 and 5], we begin to see intermediate utterances in which the women seem to be *checking for agreement* [18, 22, 49, 61-63]. It seems plausible that this shift is due in part because of the symbolic tools (Vygotsky) that the women utilize during these two phases of their discussion. One notes, for example, that Sara has properly phrased her key question as to "contain no symbols or other notation from the specific problem under consideration" (Solow, 2005, p. 10). In the planning phase then, the women mediate their discussion primarily with words (with the exception of the algebraic equation n=2k+1 in Line 6). The increased efficiency that is gained in recording ideas on paper symbolically (mathematically) in the engagement phase, however, presumably necessitates more verbal exchanges of clarification and checking to keep the communication effective on the part of the interlocutors.

We can subdivide the engagement phase into cycles. The first cycle, represented by Lines 17-41, appears to be a productive mathematical flurry. From the planning stage, Sara seems to move fluidly and without hesitation to engagement by squaring the quantity *2k+1* [17]. After confirming with Lisa what she is doing and fixing her algebraic errors [18-19, 22-24], Sara next easily determines that a two can be factored from the first two terms of the resulting trinomial, but not from the third term [27]. Lisa quickly recognizes that Sara's reasoning leads to the expression being an odd number [28]. If the transcript were to end here, one might assume that the women proceeded to put the finishing touches on the proof with the same ease with which they produced its fundamental building blocks. But whereas the ideas seemed to flow fast and furiously early, the women now appear at a loss for how they are going to "show" their ideas [31, 32]. In the second [42-83] and third [84-118] cycle of the engagement phase [29-118] the women recycle through the ideas suggested during the first cycle (see Table 4). The third cycle ends similarly to the first, with Lisa stating "but I just don't know how to go from this" [117]. On the surface, the second and third cycles seem to have accomplished little in terms of moving the proof forward. Let us examine this more closely.

Communication is a patterned activity. We are able to participate in various forms of communication because we are familiar with the routines associated with them. Take the basic communication of greeting. The act of greeting follows a routine consisting of the elements of *when* and *how.* We must understand the context (when it is appropriate to greet someone or respond to another's greeting) and we must also understand how to do it (what words to use or gestures to make). In our everyday lives, greeting is a more or less automated routine. When a situation is less than familiar, however, one is often left searching for a routine. Sfard (2008) states:

> In situations that do not automatically evoke standard routines, an ad hoc pattern would often settle in from the very first exchange. This is particularly true of educational settings. There is a salient rhythm to interactions involving newcomers to a discourse, trying to become its full-fledged participants. (p. 197)

Although ad hoc, the patterns of newcomers "are made possible by certain standard discursive patterns, already known" to them. Sara and Lisa obviously rely on an algebraic routine (see Figure 11), with which they are no doubt familiar, to keep their discourse alive. They know both when and how to square and factor the algebraic expressions. They also recognize in their resulting algebraic expressions the form of an odd number. But it is here that the familiar algebraic routine no longer serves them in advancing the proof. They get stuck each time because they are not well-rehearsed in the next routine. They realize neither that it is a good time (when), nor how to rename $2k^2 + 2k$ as a new integer. Yet while the structure of their routine remains fundamentally the same, each cycle brings about modifications. In the second cycle, for example, we see the women support each of their algebraic maneuvers with justification. In the third cycle, we see them consider for the first time how working from a different direction might be advantageous. We see then, a change in what the women are saying and doing. Sfard notes that "this kind of change is typical of interactions involving participants who are in the process of individualizing a discourse" (p. 199). In this example, it is almost as if when the old and familiar routine comes up short, the learners tweak it and try it again. To learn is to become ever more fluent in a discourse. We see, in this example, a handful of successive approximations of the perhaps thousands needed to become fluent. Let us now consider other aspects of the discourse.

Effectiveness of Communication and Productivity

In Line 84 of the transcript, Lisa suggests that they start at the bottom and "somehow get to that" (that an even plus 1 is odd). The focus of the discussion, however continues to revolve around demonstrating even and odd for various terms [93, 95, 103, 105]. When she re-suggests the idea in Line 119 the emphasis of the discussion switches from the mathematical contents of the proof to strategy. The discussion soon ends due to time constraints. A cursory read of the transcript might have suggested a different ending. After all, the preoccupational analysis gave a snapshot of an exclusively interpersonal dialogue. The women frequently use the plural pronouns *we* and *our* in their discussion [e.g., 2, 10, 12, 42]. They each demonstrate care and encouragement for the other throughout the process [13, 29, 51-52]. As mentioned previously, they make use of the forward-backward strategy and they easily produce the fundamental mathematical pieces needed to complete the proof. So why did the women not experience more success in bringing their proof to conclusion? Would the path of "reversing" their work, suggested by Lisa in the last phase of the transcript, have proved productive?

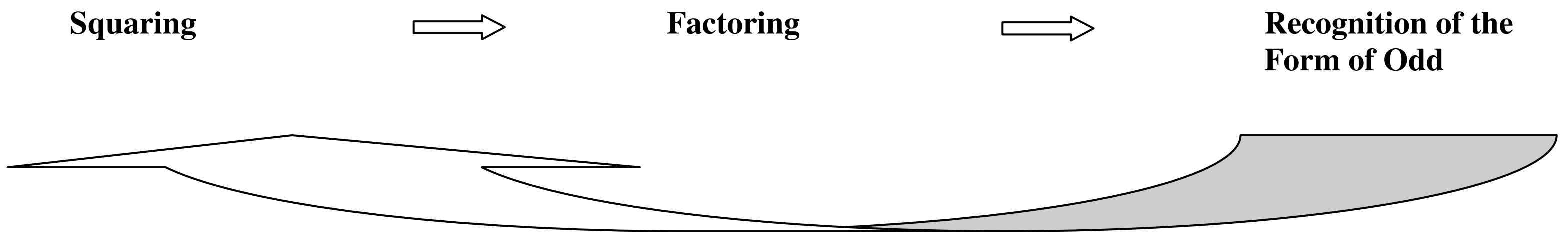


Figure 11. Ad hoc routine course of action in Excerpt B.

Table 4

Engagement Phase as Multiple Cycles of the Same Ad Hoc Course of Action

Cycle	*1* *Lines 17-41*	*2* *Lines 42-83*	3 Lines 84-118
Squaring	[17] S: N squared equals. . . . . . [22] L: Was it 4k squared plus 4k plus 1?	[42] S: So we're going to square both sides and that's just math oh just algebra [54] S: So if we square this and n squared is equal [55]L: 4 k squared plus 4k plus one [58] S: And this is algebra and this is definition and then. . .	
Factoring	[27] S: You can take a two out of the first term. Showing that one is even. You can take a two out of the second term showing that one is even. You can't take a two out of the last one.	[60] S: Okay good, we're four for four. . alright, so we can show that 4k squared is equal to 2 (2k) squared (Lisa choruses in on the 2 (2k) squared) showing that it's even, definition of even number [64] S: So same thing with the next one 4k is equal to two times 2k which is the definition of even . . . Mmm kay, Can you say that one is an odd number?	[103] S: No, no, no No watch You can pull a 2k out of both of those terms.
Recognition of the Form of Odd	[28] L: Right, so that would make it odd.	[81]: S Yeah, So if you're using this definition . . .an even plus one is odd, is an odd answer, do we have to prove that?	[110] L: Yea cause like you could divide that by 2 and it's still going to be a remainder. It can't be even. Is there like a definition that says if there's a remainder its going to be odd?

The conversation is characterized by interpersonal utterances by both interlocutors. How do the utterances of each shape its direction and success? Focal analysis reveals that the women's contributions are very different in their nature. Sara appears to take the lead, so to speak, from early on and throughout much of the conversation. She offers the key question [4]; abstract answer [6]; and the hypothesis and conclusion [11]. She also is the first to articulate the logical arguments for the body of the proof [17, 27] and their justifications [42, 58, 60, 64, 66, 79, 81]. Indeed, Sara might be described as managing the mathematical *content* of this discussion. Lisa's contributions, on the other hand, are mainly in the form of affirmations [5, 7, 43, 47, 59, 61] and questions. Some of Lisa's questions serve the purpose of clarification [e.g., 18, 22]. Many other questions relate directly to the necessary *approach* for successful completion of the proof [2, 10, 12, 20, 32, 84, 87, 90, 119]. Let us examine each of the women's contributions more closely, as informed by the focal and preoccupational analyses.

To begin their work on the proof, Sara chooses an appropriately broad key question [4] "How do I show *a number* is odd?" (as opposed to How do I show n^2 is odd?). Yet rather than keeping her abstract answer similarly broad (i.e., show the number is equal to the sum of twice an integer and one), Sara introduces variables indicating that the abstract answer "would be n=2k+1" [6]. Lisa affirms this [7] and both write $n=2k+1$ down as the abstract answer (see handouts in Figure 9 and Figure 10). Arguably, everything indicates that, from the women's perspective the conversation is proceeding effectively. But is it? We can surmise that the communicative purpose of their exchange was to *get* what it means for a number to be odd, and that the exchange was successful for both. Their choice for the abstract answer, however, may be the first hint in the conversation of an ambiguity surrounding *what* they need to prove (conclusion or B). After all, the goal is to show that the *square* of an odd number is odd. We can almost forecast the issue that using $n=2k+1$ as the abstract answer presents, as the hypothesis (given) was presented in the same form. It is noted here that for an expert such casual and potentially overlapping notation $(n=2k+1)$ would rarely be problematic and would likely be more than sufficient for guiding a successful line of thinking. Lisa's question in Line 10a, "Do we need an applied answer?" provides a window of opportunity to avert the issue that has been unknowingly set up in Lines 4-7. However, Lisa immediately seems to dismiss the importance of her question by stating, "That's the same thing" [10b].

It is unclear what Lisa is describing by her use of the demonstrative pronoun "that's." Does she intend that the terms *abstract answer* and *applied answer* are synonymous? Does she think that in this particular instance the abstract answer and the applied answer happen to be the same? Or does she recognize the distinction between the two, but utilize "that's" to reference the concept of odd? Sara ostensibly understands what Lisa means, as she replies "Yea" [11a]. She then proceeds to also make use of her own "that's": "That's just n squared is an odd integer" [11a]. Again, it is uncertain what Sara intends by the utterance. Is she referencing a permeating concept of odd? Or, is she revising her original abstract answer? Rather than illuminate the issue, her written work adds to the uncertainty. On her handout (see Figure 9), she has B written twice: at the top of the page as $n=2k+1$ (the same as her Abstract Answer written two lines above it) and as n^2 *is odd* at the bottom of the page. Sfard (2008) states that the effectiveness of mathematical communication is "constantly being threatened by the circularity of the process of development of mathematical discourse and by the pervasive vagueness as to the nature of its objects" (p. 161). Sfard calls the resulting breakdown in communication *commognitive conflict*. Specifically, it is "the encounter between interlocutors who use the same mathematical signifiers (words or written symbols) in different ways or perform the same mathematical tasks according to different rules" (p. 161). Commognitive conflict can be both interpersonal and intrapersonal in form. In the passage just discussed, we see hints of it on both planes.

Such a microscopic look at two lines [10,11] of transcript may seem, at first, pedantic. But the unrealized potential of this particular interchange becomes glaring when viewed in light of the discourse that unfolds. (Recall, after all, that one task of the discursive researcher is uncovering breaches in communication.) With all of the mathematical content in place to complete the proof [17-28], we saw that the women generally come to an impasse [29-118]. While we cannot be certain, it seems that a more explicit (both oral and written forms) interchange at this point, could have facilitated an overall different and more successful outcome of the entire discourse. I surmise next how this might have occurred.

That's the same thing [10] Lisa's utterance suggests a comparison (specifically, an equivalence). Lisa could have increased her explicitness in two important ways. First, she could have replaced "that's" with the noun (mathematical object) that she was intending to describe. Secondly, she could have made overt the "thing" to which "that" is being compared. Moreover, given Lisa's original statement, it certainly would have been behooving of Sara to verify her interpretation against Lisa's intention. Either process of explication may have provided the entry point that the women needed to revisit more closely the ideas of abstract and applied answers.

Whereas an abstract answer is broad in nature, the applied answer is specific to the problem at hand and makes use of appropriate notation. Thus, a fitting abstract answer in words reads, “Show that the square of a number is twice an integer plus one.” An applied answer of “$n^2 = 2l + 1$ where l is an integer” uses the notation of the proposition and the definition of odd number. When we examine the transcript and the work on the women’s notebook paper, it becomes easy to conjecture how having such an applied answer may have helped them. For example, Line 81 of the transcript and the last three lines on Sara’s notebook paper (see Figure 9) seem to suggest that Sara wants to “prove” the definition of odd (prove that adding one makes it odd) rather than *use* the definition to justify that n^2 is odd. If Sara would have written $n^2 = 2l + 1$ on Line 10 (and similarly Lisa) in place of n^2 *is odd,* she may have recognized that in her work she had precisely what she needed—twice an integer plus one—to prove the square of an odd number odd.

The women may not have reached a desirable end even if they had been more explicit in their utterances [10-11] and as a result revisited their abstract and applied answers. After all, they still needed to identify $2k^2 + 2k$ as the integer l in order to complete the proof. Nonetheless, the question that Lisa proposed (“Do we need our applied answer?”) represented an opportunity not fully exploited. This may not have been the only time a contribution’s potential went unrealized. Preoccupational analysis reveals that Lisa questions the pair’s *approach/process* nine different times within the discourse. (Note Lines 2,10,12,20,32b,65,84, 87,90,110,119 in the transcript]. Let us look briefly at a select few of these utterances and consider the effects they had or could have had on the discourse.

1. *What's our key question?* [2] By posing this question, Lisa provides a focus for the discourse from the start.
2. *Are we working from the bottom or the top first? I don't think it matters so which one?* [12] This question, posed by Lisa, provides yet another entry point for the women to examine closely their progress and consider how their work will unfold. Lisa is somewhat dismissive of the importance of what she has posed stating, "I don't think it matters." Moreover, it is likely that this comment does little to encourage/invite Sara to consider analytically each possibility. In the end, the women go off Sara's intuition—she "feel[s] like it goes somewhere from the top, cause you can't really do anything from the bottom." In practice, the forward-backward method is an accordion-like motion, in which one tries to connect the assertions derived from the hypothesis (A) with statements that demonstrate that the conclusion (B) is true. In Lines [29-118] the link between hypothesis and conclusion is never made. Had Lisa and likewise Sara considered more carefully the implications of Lisa's question, the result may have been different. At the very least, they could have committed to working *both* forward and backward, even if they chose to begin working from the hypothesis. However, if they had taken pause to work from the bottom, they could potentially have narrowed the impending chasm. Both had B (the conclusion) written as n^2 *is odd* (see Figure 9 and Figure 10). The next step in the backward method would have been to ask "How do I show n^2 is odd?" (the original key question). The answer, call it B1, is to show $n^2 = 2l + 1$ where l is an integer—a step crucial for closing the gap in their work. With more attention to Lisa's question [12] the course of the women

might have righted here, irrespective of their arguably inexact treatment of the abstract/applied answer [6-11].

3. *Should we prove n is even first and then if you add a one it's odd? But that would be backtracking that really doesn't make any sense* [20]. We get a sense from reading the transcript that as Sara pushes the mathematical content and algebraic manipulations [11, 17, 27] forward, Lisa's attention is simultaneously split. She follows/checks/confirms what Sara [18, 22] is saying, but also deliberately considers the overall proof strategy/process. Much like in line 12 though, Lisa seems to dismiss her own question [20] stating that it "really doesn't make any sense." Sara responds with a "yea." Translated literally, Lisa's *specific* proposition to prove n even does not have much potential since n is given as an odd number in the hypothesis. Because neither of the women chose to pursue the question further, it had little consequence for the overall discourse. But could it have? Was Lisa truly suggesting that they prove the given n even? Alternatively, was the issue just one of overlapping language? Did Lisa intend that they should work to show they had an even number plus one to prove odd? Had the posed question generated more dialogue than it did, had the women taken the time to clarify—to tease out—Lisa's thinking, perhaps they could have thwarted the difficulty of "showing odd" that crops up later in the transcript [31,32].
4. *Well like how did we do the first one because?* [32b] In this utterance Lisa suggests that the process ("how") utilized in a proof (Problem 3.11, see Figure 8) completed in a whole-group discussion earlier in the seminar may be of use in their current situation. Unfortunately, the audio recording surrounding this

utterance was inaudible, so the nature/productivity of the discussion surrounding this suggestion cannot be fully determined. It should be noted, however, that the women's written work for Problem 3.11 might have provided them with some much needed clues to close a gap, symbolically represented by a downward pointing vertical arrow on Sara's notebook paper (see Figure 9) in their argumentation. In the whole-group proof for Problem 3.11, the integer *m* was substituted for the sum of integers *k* and *l* to produce the desired form of an even number. A similar strategy, substituting integer *l* for $2k^2 + 2k$ where *k* is an integer proves helpful in Problem 3.9.

5. *Maybe we should have did this from down here. You know what I mean, we should of said this is from the bottom up, yea, I think we should have* [119]. From line 119 forward, the discourse is filled with demonstrative ("this," "that") and subjective ("it") pronouns, making it difficult to fully decipher what Lisa intends when she says she wants to reverse [121] the process. Initially, Sara asks Lisa to "show me what you mean" [120]. Successive interchanges do not indicate that Sara perceives a breech in communication. She seems to "understand" Lisa [126,128,130]. Nonetheless, Sara has admitted difficulty explaining the reversal process in her own words [140]. Perhaps the best signifier of the women's newly arrived at intention is the arrow that Lisa has drawn on her notebook paper (see Figure 10). It appears that they are beginning to connect their own work with the *form* of an odd number—twice an integer plus one. Their interchanges suggest though, that they want to conclude with line 2 of their work. The issue of applied/abstract answer [10b] is still unresolved at this point. Interestingly, lines

87 and 90 of the transcript point to what might have been a growing recognition on Lisa's part of this issue. Because of the practicality of time, the women's discussion effectively ends here, with them trying to explain how they are going to "reverse their work."

Summary

This excerpt has proven an interesting test case in student discourse on mathematical proof. First, we saw the learners relying on a routine consisting of familiar algebraic discursive patterns to navigate a new discourse (proof). They relied on the structure of the ad hoc routine to keep the discourse moving, making minor modifications to the routine each time they cycled through it. It was suggested that this is a microscopic snapshot of learning mathematical proof—a series of countless changes both infinitesimal and incremental in novice discourse.

A close examination of the discourse also showed that the appropriate introduction and manipulation of mathematical content in the discourse was insufficient in leading the women in this excerpt to a mathematically satisfying conclusion (in the allotted time). This was in spite of the fact that the women fostered an exclusively interpersonal discourse, characterized by inclusive language and encouragement. So, why, with seemingly requisite components in place (mathematical content and supportive discourse exchanges), were the women unable to reach a satisfactory ending? First, we see indication of commognitive conflict. As they work to develop their mathematical discourse we see the women, at both intra- and interpersonal levels, become entangled in the vagueness of the mathematical objects (abstract answer, applied answer, odd). Moreover, the analysis points not in the direction of what *was* attended to in the discourse, but rather what was left *unattended.* Lisa posed frequent questions related to *approaching* the proof. However, the women gave the questions cursory attention at best. The analysis suggests that these questions related to *approach* may actually have been the much-needed discursive entry points that the women needed to refine, focus, and make explicit their discourse.

Excerpt C

Overview

Excerpt C is from the third seminar on proof. In its entirety, the conversation lasts nearly forty-five minutes. The interlocutors, working on Problem 3.17 from the Solow text (see Figure 12) are two freshman men: Jacob and Patrick**.** Late in the transcript [222], a non-participant joins the group discussion. Appendix G omits this portion of transcript as the findings presented next focus on the interchanges of Jacob and Patrick prior to the entrance of the non-participant.

Periods of off-task discussion related to things including, but not limited to, the weather and another class they are both taking generously pepper the entire conversation. In the transcript, each off-task passage is condensed into a single line noting as much. This preserves the confidentiality of the participants and lends expediency to the reader's task. The reader is invited at this time to read the whole transcript. However, for the convenience of the reader, passages especially significant to the analysis are included in the text below.

Problem 3.17
Use the definition of an isosceles triangle to prove that if the right triangle *UVW* with sides of lengths u and v and hypotenuse of length w satisfies $\sin(U) = \sqrt{u/2v}$, then the triangle *UVW* is isosceles.

Figure 12. Problem 3.17 from Daniel Solow's *How to Read and Do Proofs* (4th ed.).

Off-task discussion is not the primary focus of the upcoming discussion. The presence of off-task discussion, however, is not insignificant and it will be taken up as it relates to the organic whole of the findings. Two points about off-task discussion do bear mentioning here. First, as first semester freshman sharing the same major these off-task interludes likely allowed the men to get to know each other on a more personal/informal level. Secondly, the reader of the transcript may question to what extent the off-task discussion had an interruptive effect on potentially mathematically productive points in the conversation [35,67,83,111].

There are two broad phases to the conversation under examination. Miscommunication regarding the hypothesis and conclusion characterizes the first phase of the conversation [1-196] and prolonged algebraic embroilment dominates the second phase [197-345]. The upcoming analysis is a close examination of the first phase only. It is not so much the misinterpretation over the hypothesis and conclusion that is of interest, as it is the men's persistent failure to communicate about it.

Persistent Ineffective Communication

The excerpt starts with Jacob reading the beginning of Problem 3.17 [1] followed by a brief interlude in which the men adjust to the digital recorder and generally settle in [2-4]. Patrick then proceeds to read the problem [5]. While he reads more of the problem aloud than Jacob did, it is of interest that he does not read the conclusion *then the triangle UVW is isosceles* aloud. He then quickly concludes, ostensibly because of his limited background in trigonometry, “I have no idea what I am doing” [7]. After a short exchange about the mathematical classification of the problem’s content and Patrick’s familiarity with it [8-11], a period off-task discussion [13-19], and Jacob’s offer to give Patrick a “quick Trig lesson”[20], the dialogue in Figure 13 takes place.

22	Jacob	We’re not worried about that in this instance. And our *A* is triangle *UVW* is isosceles.
23	Patrick	So you’re writing it as two proofs then?
24	Jacob	Just one
25	Patrick	Cause *A* is that it’s a right triangle *and* satisfies this equation and the *B* is that it is isosceles. Trying to figure out that it’s isosceles based on it’s satisfaction of that equation
26	Jacob	Oh really
27	Patrick	and the fact that it’s right
28	Jacob	Oh, alright
29	Patrick	So you’re writing it as two separate proofs. . . .you have like an *A* and an *A* sub B.

Figure 13. Dialogue between Jacob and Patrick.

The men are using *A* and *B* here to represent the hypothesis and conclusion respectively. This is the practice introduced in the Solow text and reinforced in the seminar sessions. It is quickly apparent that Jacob has mistaken the conclusion, that *triangle UVW is isosceles*, for the hypothesis [22]. Patrick’s utterance [25], on the other hand, demonstrates that he has a firm grasp on what they can assume to be true (hypothesis) and what they need to demonstrate as true (conclusion). What is interesting, however, is Patrick’s response when Jacob states “*A* is triangle *UVW* is isosceles.” He does not immediately correct Jacob. Rather, he asks, “So you’re writing it as two proofs then?”[23]. Two interpretations seem likely. Perhaps the question was Patrick’s indirect, albeit polite, attempt to determine the error in Jacob's thinking. But when Jacob replies that he is just writing it as one proof, it is almost as if Patrick *believes* that the proof could be handled as two separate proofs with "like an *A* and an *A* sub B" [29]. Writing two separate proofs, after all, was exclusively Patrick's idea. In light of his earlier comment that he did not know what he was doing [7], the question in response to Jacob's assertion takes on a deference-like quality. It is almost as if because Jacob knows calculus, and he does not, Patrick gives him the benefit of the doubt, which, in this case, is unjustified.

Jacob's brief replies of "oh really" [26] and "oh, alright"[28] provide little evidence that his thinking has changed**,** despite Patrick's rather textbook-like explanation of the hypothesis and conclusion [25, 27]. As the transcript progresses, it is evident that Jacob still mistakes the conclusion for the hypothesis [64,92] even though Patrick continues to state it correctly[61] (see Figure 14). Following Sfard, it is instructive if we look for breakdowns in the communication. We have seen in Excerpts A and B breakdown in the lacunas of the conversations—in what is not said. Thus, in this excerpt, it is confounding that the very idea the pair most need to prevent and eventually correct what becomes an ill-fated course *jointly* pursued, is not only stated, but *explicitly* stated by one of the interlocutors [25]. What went wrong?

60	Jacob	a)That would be bad, b) so in order for this to be true
61	Patrick	It needs to satisfy that equation and be right and then be isosceles
62	Jacob	*W* has to equal 2v Right?
63	Patrick	Right
64	Jacob	It has to satisfy that equation and be right, alright. . . . Oh wait, if this is right and it's isosceles. This has to be 45 degrees, this has to be 45 degrees. . . Where am I going with this? (Long pause) Oh wait, square root, where did the square root come from? Pythagorean theorem maybe? *U* squared plus *v* squared equals *w* squared. And if it's isosceles *a* will, or *b* will equal *u* so it will be two *u* squared or should we go with *v* squared?
65	Patrick	2 *u* squared equals *w* squared.

91	Patrick	So, 2 *v* squared plus *w* squared or 2 *v* squared equals *w* squared so (pause)
92	Jacob	What else do we know about isosceles triangles? (long pause)

Figure 14. A series of utterances between Jacob and Patrick related to the hypothesis and conclusion.

The discussions of the previous two excerpts have explored in depth the interlocutors' attention in their discourse, in terms of both quantity and quality, to the key question and abstract answer. That angle of analysis will not dominate the discussion of this excerpt. However, it would be remiss not to mention that the men fail to employ these strategies on their own, and to speculate how differently the conversation may have proceeded had they attempted to do so. Posing a key question may have provided Jacob with the framework that he needed to sort out all that Patrick was explaining [25, 27]. At the very least, coming up with a mutual key question may have provided Patrick with an unambiguous means of addressing his partner's misunderstanding.

We have already seen that, early in the conversation, right in the midst of providing a lucid explanation of the given and what they need to prove [25], Patrick additionally entertains the idea that if broken into two separate proofs, Jacob's assertion [22] is plausible. The transcript does not provide any evidence that allows an interpretation regarding the extent to which Jacob was engaging Patrick's utterances. If Jacob was engaged to the fullest extent possible, he might have detected a contradiction in Patrick's thinking, leading him to re-examine his own. What is more likely is that Patrick's digressions about two proofs lessened the impact of an otherwise succinct explanation [25] for Jacob. The ineffective communication in this initial interchange [22-29] carries through the first phase of the conversation. .

After a quick, albeit technically imperfect, trigonometry lesson [42-51] and some off-task discussion [52-59], Patrick makes the following utterance, once again indicating that he has correctly determined the hypothesis and conclusion: "It needs to satisfy that equation [$\sin(U) = \sqrt{u/2v}$] and be right and then be isosceles" [61]. Yet, three lines later in the transcript he does not attempt to correct Jacob's blatant and repeated assumption that the triangle *is* isosceles [64]. Moreover, one line after that in the transcript, Patrick makes the assertion "2 *u* squared equals *w* squared" [65], which requires the legs of the right triangle to be of equal length. The reader will also note that Patrick, like Jacob, labeled the right triangle on his paper with $45°$ angles, thereby making it isosceles (see Figure 15 and Figure 16). Lines 91-92 represent yet another incongruence between what Patrick seemingly understands [25, 27, 61] (the correct hypothesis and conclusion) and his participation in the joint work. His statement "2 *v* squared equals *w* squared" [91] again assumes the triangle isosceles and he does not intervene as Jacob continues to look for ways to move forward from that same incorrect assumption [92].

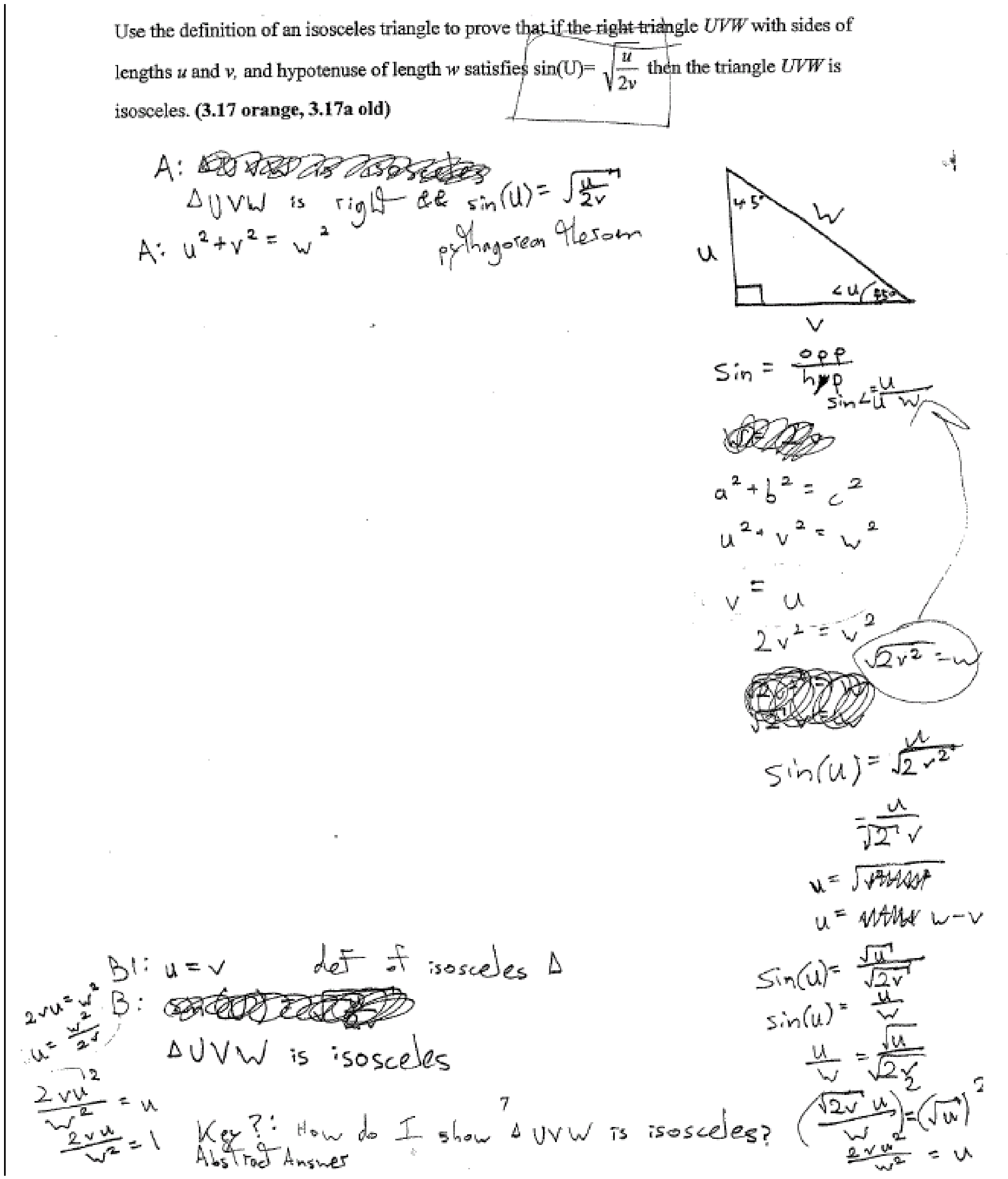


Figure 15. Jacob's work on Problem 3.17.

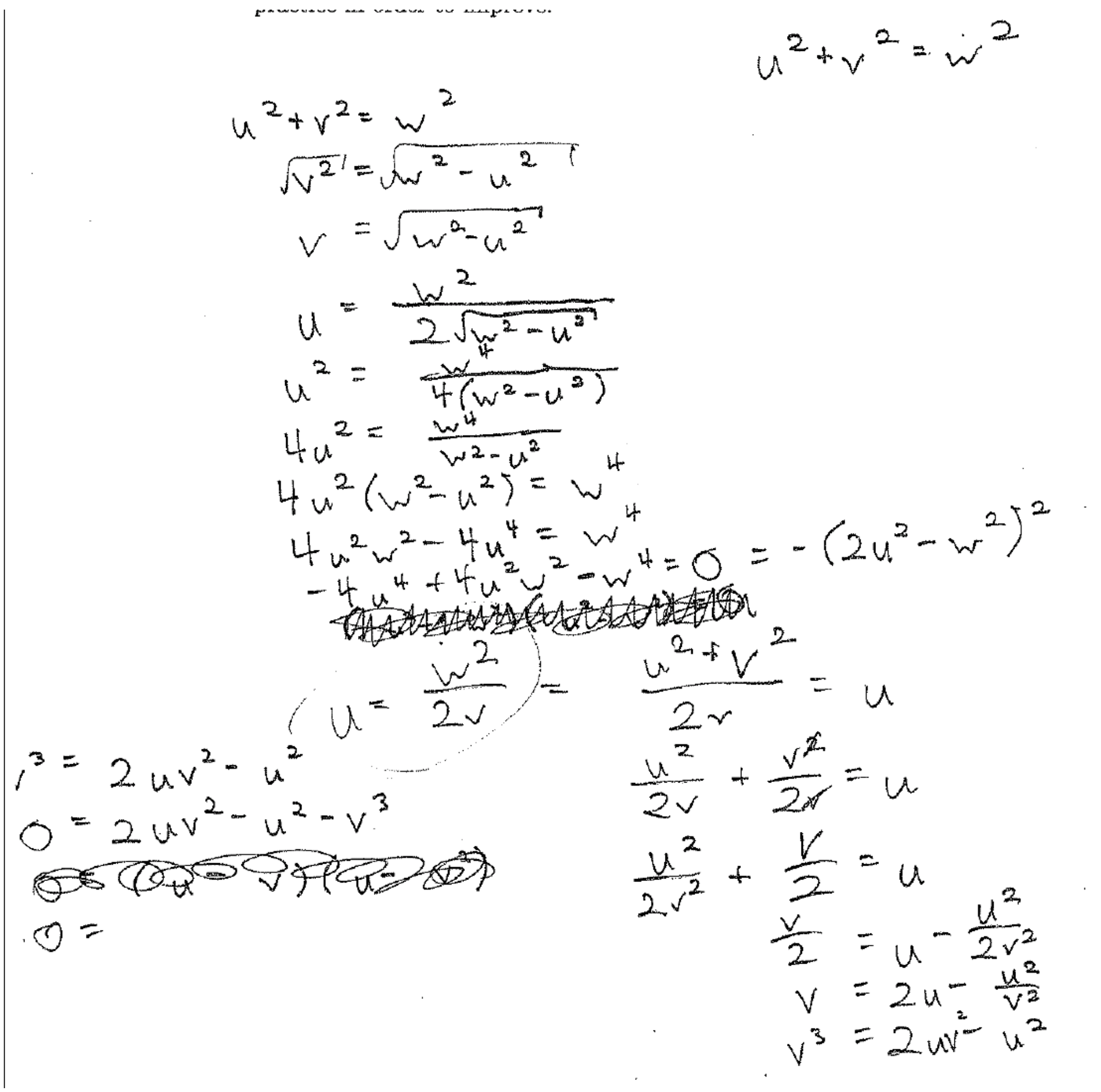


Figure 15. Jacob's work on Problem 3.17 (continuation).

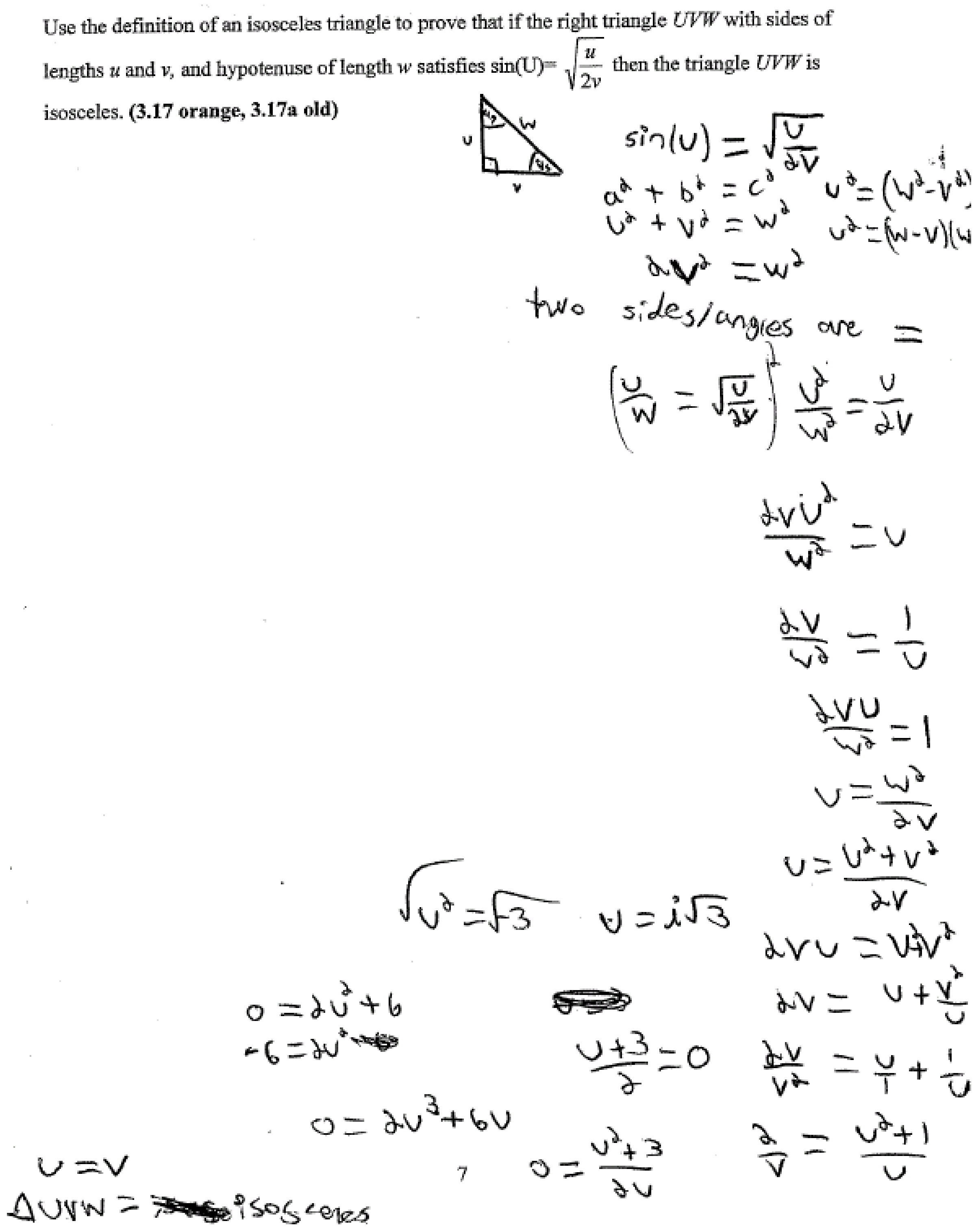


Figure 16. Patrick's work on Problem 3.17.

If it were not for the next part of the transcript (see Figure 17), the reader might justifiably question whether Patrick comes to believe that they *can* assume that the triangle is isosceles. However, I soon join the men to see how they are progressing in their work [132]. In Line 143, Jacob begins describing specifically their work. When he states "we're assuming that it's isosceles and a right triangle" [145], I respond by questioning whether they can "assume that it's isosceles from the start" [146]. It becomes quickly evident that Patrick does in fact correctly recognize the hypothesis and conclusion. Without pause, he reiterates what he said at the beginning of the conversation: "No cause that's [the triangle is isosceles] B. We can assume it's right" [147]. Moreover, in stating "If right and that equation [$\sin(U) = \sqrt{u/2v}$] then isosceles" [152], he nearly echoes the succinct statement of the proposition that he had already made in Line 61. The conversation continues as I encourage them to form and answer a key question [154,171,173,182] and help Jacob to recognize the parts of the conditional statement in the overall problem statement [165]. Jacob, who initially admits some confusion [163], gradually demonstrates through a series of utterances a correct understanding of the hypothesis and conclusion [166, 170, 174, 185, 187, 195].

143	Jacob	So we have *sin* is opposite over hypotenuse.
144	Me	Okay.
145	Jacob	And here's side *u* and we're assuming that it's isosceles and a right triangle.
146	Me	Okay, now can you assume that it's isosceles from the start?
147	Patrick	No cause that's B. We can assume it's right. (long pause)
148	Patrick	Who's turn is it to say something?
149	Jacob	Why you all looking at me? I was just rereading the problem.
150	Me	Okay, go ahead.
151	Jacob	Um, using the definition of an isosceles triangle to prove that it is right. . . oh so we have to prove that it is right.
152	Patrick	If right and that equation then isosceles.
153	Jacob	Oh, okay, if it's right and this equation and by starting with isosceles.
154	Me	So what's our key question?
155	Jacob	How do you show a right triangle?
156	Me	Are we trying to show that the triangle is a right triangle?
157	Patrick	Isosceles.
158	Jacob	Oh so I guess isosceles?
159	Patrick	How do we show isosceles triangle?
160	Jacob	Alright.

Figure 17. Continuation of identification of hypothesis and conclusion.

The reader may have an understandable feeling of uneasiness about the productivity of the just analyzed mathematical conversation. Clearly, the analysis lacks any evidence that points to either interlocutor benefitting in any significant way in relation to the specific proof under discussion or the proof process in general. Yet Patrick clearly had the knowledge that the pair needed to move forward successfully. Why was this knowledge never effectively communicated? Where did the breakdown occur? The summary explores several considerations to these questions.

Summary

Patrick's concise explanations [25, 27, 61] of the hypothesis and conclusion of the proposition are likely to resonate with those who have expertise with the mathematical material under discussion. We see, however, that these explanations did not help Jacob realize the error of his assumption. There was clearly a breach in effective communication. Jacob responded well, though, to my interjection [154-195]. The potential for Jacob to revise his assumption existed in the interactions he had with both Patrick and me. Why the discrepancy in how Jacob responded?

It is perhaps the case that Jacob's posture for revising his thinking was more open when the dialogue was with an expert/authority figure. The implications of this possibility cannot be ignored. Sfard (2008) says, "the issue of leadership in discourse is, of course, a matter of power relations" (p. 283). If Jacob has been mostly exposed to traditional mathematics classrooms throughout his education, then he may view a teacher as having the final word.

Let us consider another possibility. Jacob was initially unable to decipher the components of the conditional on his own in reading the text. It seems probable, then, that he was no more able to decipher Patrick's summaries of the proposition (in particular Line 61). The comments did not catch Jacob's attention as particularly significant for his own learning. An analogy is perhaps in order. The transcript is to the actual discourse (and all that it embodied) as a contour relief map is to a rugged landscape. From the perspective of an expert looking down at the transcript in its entirety, Patrick's explanations [25, 27, 61] are like relief lines, representing the highest point on a contour map (the discourse). From a privileged aerial view, these comments catch our attention. But Jacob has no map in the thick of the terrain and his journey includes forays into off-task discussion intertwined with algebraic manipulations. What is more, Patrick, the very person who *is* privy to the map, discards it and goes along for the ride with Jacob rather than pointing him in the right direction. Recall that communication is a patterned activity.

Patrick's communication did not include a familiar routine for doing math or for learning in general that Jacob could latch onto. However, as a student for many years, Jacob has no doubt automated the IRE (initiate-respond-evaluate) routine. Thus, when I begin to elicit information from him, he likely recognizes the interchanges as significant to his learning. Figure 18 illustrates the pattern.

169	Me	**Initiate**	*Then*, so the then part, the B part, If A then B, is we want to be able to show that the triangle is. . .
170	Jacob	**Reply**	Isosceles.
171	Me	**Evaluate** **Initiate**	Isosceles, so your key question
172	Jacob		Oh.
173	Me	**Initiate (continue)**	Is how do I show a triangle is
174	Jacob	**Reply**	Isosceles.
175	Me	**Evaluate** **Initiate**	Isosceles. Okay, they want you to use the definition to do this proof. So what's the definition of isosceles triangle?
176	Patrick	**Reply**	The definition of isosceles triangle is . .
177	Me	**Initiate**	So if you ask the question how do I show the triangle is isosceles
178	Patrick	**Reply**	We could define isosceles
179	Me	**Evaluate** **Initiate**	Right, which is what?
180	Jacob	**Reply**	Uh, two sides are
181	Patrick	**Reply**	Equal, or 2 angles.

Figure 18. Initiate, reply, evaluate pattern in Excerpt C.

Finally, we cannot help but ask why Patrick seems to discard the map and go along with Jacob. It could be that Patrick knew what the hypothesis and conclusion were from the proposition, but did not understand them to be unique immutable entities. Rather he proposes that it was perhaps possible to assume an isosceles triangle if they came up with two proofs [23, 29]. From the beginning, however, Patrick expresses doubt about his own ability to work on the proof because of his limited trigonometry background [7]. While he clearly understands where the proof begins and where it ends, perhaps he feels the need to rely on Jacob for the intermediary steps. His lack of knowledge, then, renders him, at least in his own mind, less authoritative in working on the body of the proof. He says as much. When I stop by to assist them, Patrick comments, "It would probably help if I had any expertise in this area, but Jacob's the only one that has taken trig" [137]. Yackel and Cobb (1996) have shown an evolution in children's understanding of what counts as an acceptable mathematical explanation and justification—the sociomathematical norms of an inquiry-based classroom. They found:

> A preliminary step in children's developing understanding of what constitutes an acceptable mathematical explanation is that they understand that the basis for their actions should be mathematical rather than status-based. Developing this preliminary understanding is not a trivial matter, especially since children are often socialized in school to rely on social cues for evaluation and on authority-based rationales. (p. 467)

There was a definite mathematical basis to Patrick's contributions to the discourse regarding the hypothesis and conclusion of the proposition. However, the analysis additionally reveals the precedence of a competing social basis for mathematical explanation and justification—a comparison of transcripts. Thus, the current example illustrates a complex interplay of sociomathematical norms in paired-discourse of undergraduate mathematics majors.

Excerpt D

Overview

Excerpt D is from the fourth session on mathematical proof. It is the paired-discourse between Lisa and Patrick, first introduced in Excerpt B and Excerpt C respectively. The conversation, slightly less than 22 minutes in length, revolves around Problem 3.18 (see Figure 19). The reader will note that Problem 3.18 contains the same proposition as the one from Problem 3.17 discussed in Excerpt C. However, Problem 3.18 directs the students to use Proposition 1 (see also Figure 19), rather than the definition of isosceles triangle, to complete the proof. Some additional background context is in order before the analysis begins.

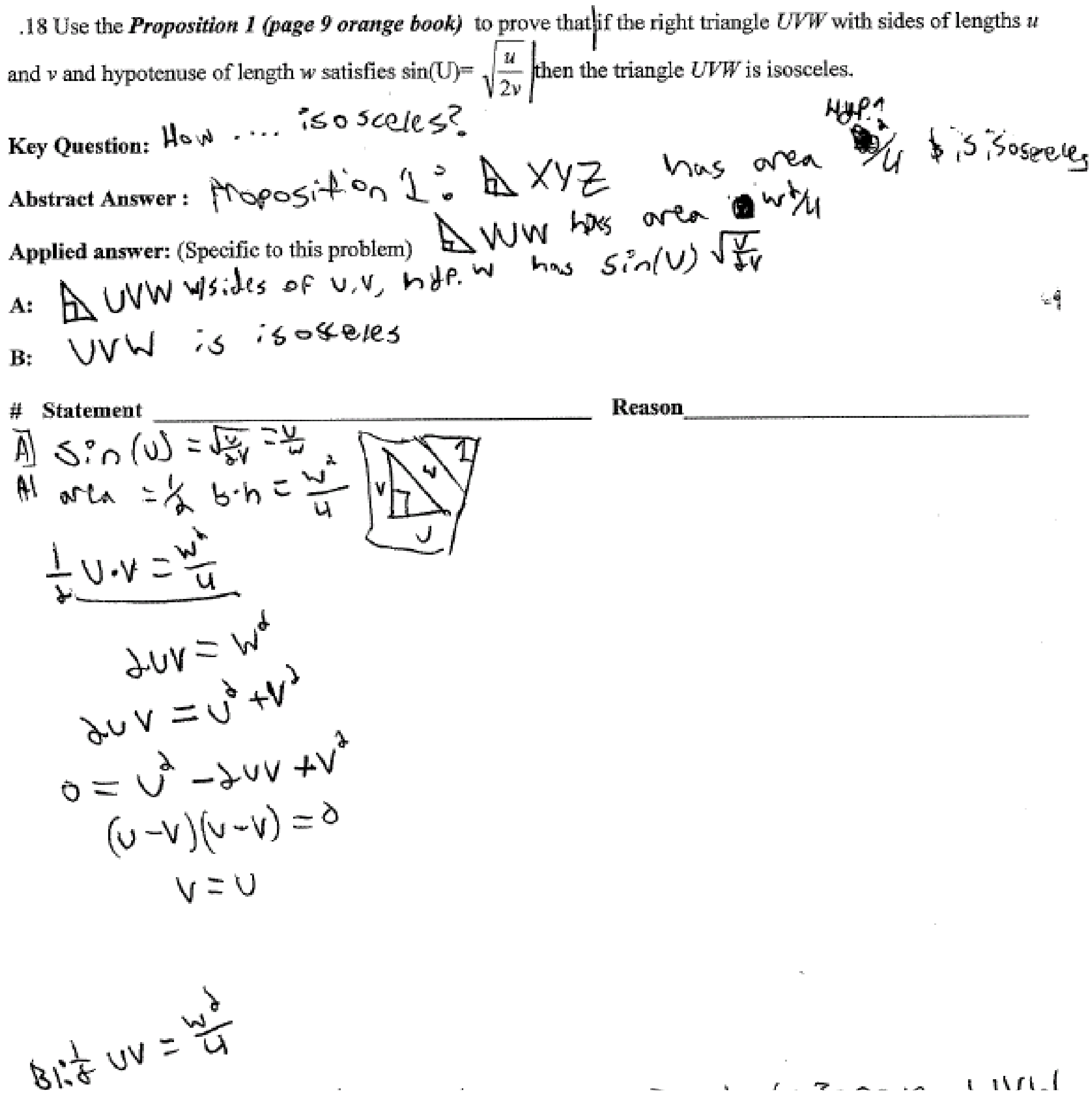
First, unlike previous sessions, the accompanying handout for the fourth session included the typed problem along with specific spaces for students to write the key question, abstract answer, applied answer, A (hypothesis), B (conclusion), the analysis of the proof, and a condensed proof. These modifications were a purposeful instructional response to the lack of these elements in the discourse of previous weeks. Second, before the students got into small groups I strongly encouraged them to conclude their proof analysis with a condensed proof. During the previous seminar, students had examined four different versions of condensed proofs for a single proposition in the Solow text. They also went over a handout with 15 tips for writing mathematical proofs. During the fourth session, just prior to the paired-discourse in this excerpt, the students received a handout with an already completed proof analysis for Problem 3.17 and as a whole group developed a related condensed proof. Finally, the reader will likely note greater instructor intervention in this excerpt than in Excerpts A, B, and C. The implications of this comparison and others will be made in the summary of this excerpt and in Chapter 5. The reader is invited, at this time, to read the transcript in its entirety (Appendix H) and examine the student work (Figure 20 and Figure 21) before reading on.

Problem 3.18 Use Proposition 1 to prove that if the right triangle *UVW* with sides of lengths u and v and hypotenuse of length w satisfies sin(U)= $\sqrt{\frac{u}{2v}}$ then the triangle *UVW* is isosceles.

Proposition 1 If the right triangle XYZ with sides of lengths x and y and hypotenuse of length z has an area of $\frac{z^2}{4}$, then the triangle *XYZ* is isosceles.

Converse of Proposition 1 (Problem 2.23) If right triangle XYZ is isosceles, then the area of the triangle is $\frac{z^2}{4}$.

Figure 19. Problem 3.18, Proposition 1 and the converse of Proposition 1 from Daniel Solow's *How to Read and Do Proofs* (4th ed.).


.18 Use the ***Proposition 1 (page 9 orange book)*** to prove that if the right triangle UVW with sides of lengths u and v and hypotenuse of length w satisfies $\sin(U)=\sqrt{\frac{u}{2v}}$ then the triangle UVW is isosceles.

Key Question: How .... isosceles?

Abstract Answer : Proposition 1: △XYZ has area $w^2/4$ is isosceles

Applied answer: (Specific to this problem) △VUW has area $w^2/4$ has $\sin(U)\sqrt{\frac{v}{2v}}$

A: △UVW w/sides of u, v, hyp. w

B: UVW is isoseles

 # Statement ______ **Reason** ______

A) $\sin(U) = \sqrt{\frac{v}{2v}} = \frac{v}{w}$

A1 area $= \frac{1}{2} b \cdot h = \frac{w^2}{4}$

$\frac{1}{2} u \cdot v = \frac{w^2}{4}$

$2uv = w^2$

$2uv = u^2 + v^2$

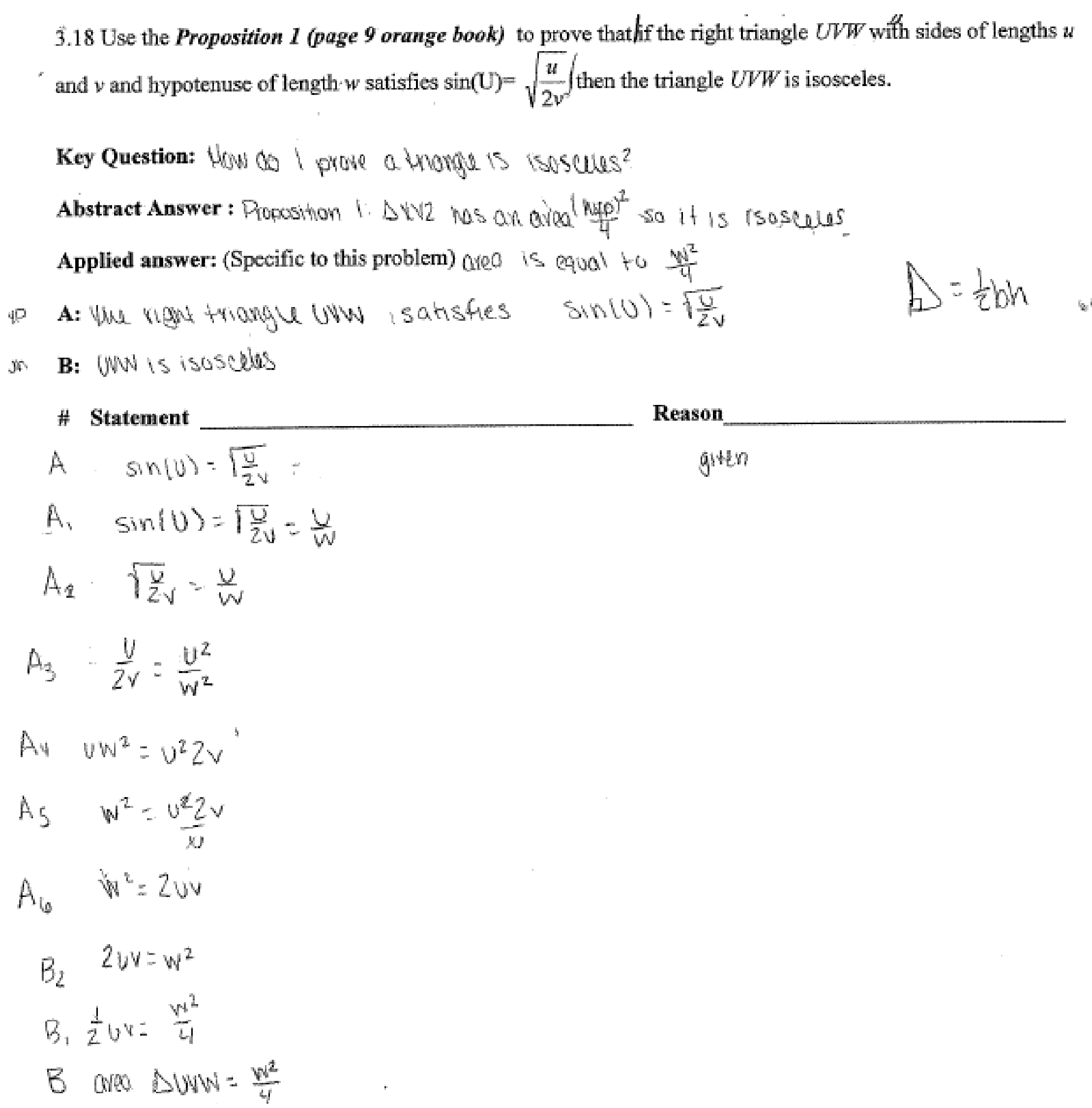
$0 = u^2 - 2uv + v^2$

$(u-v)(u-v) = 0$

$v = u$

B1: $\frac{1}{2} uv = \frac{w^2}{4}$

Figure 20. Patrick's work on Problem 3.18.



3.18 Use the ***Proposition 1 (page 9 orange book)*** to prove that if the right triangle *UVW* with sides of lengths *u* and *v* and hypotenuse of length *w* satisfies $\sin(U)=\sqrt{\frac{u}{2v}}$ then the triangle *UVW* is isosceles.

Key Question: How do I prove a triangle is isosceles?

Abstract Answer : Proposition 1: ΔXYZ has an area $\frac{(hyp)^2}{4}$ so it is isosceles

Applied answer: (Specific to this problem) area is equal to $\frac{w^2}{4}$

$\Delta = \frac{1}{2}bh$

A: the right triangle UVW satisfies $\sin(U)=\sqrt{\frac{u}{2v}}$

B: UVW is isosceles

#	Statement	Reason
A	$\sin(U)=\sqrt{\frac{u}{2v}}$	given
A_1	$\sin(U)=\sqrt{\frac{u}{2v}}=\frac{u}{w}$	
A_2	$\sqrt{\frac{u}{2v}}=\frac{u}{w}$	
A_3	$\frac{u}{2v}=\frac{u^2}{w^2}$	
A_4	$uw^2=u^2 2v$	
A_5	$w^2=\frac{u^2 2v}{u}$	
A_6	$w^2=2uv$	
B_2	$2uv=w^2$	
B_1	$\frac{1}{2}uv=\frac{w^2}{4}$	
B	area $\Delta UVW=\frac{w^2}{4}$	

Figure 21. Lisa's work on Problem 3.18.

The transcript for Excerpt D can be broken into three phases. The first phase [1-90] serves as an organization phase, where Lisa and Patrick attempt to make sense of the proposition. Work on the proof analysis generally dominates the second phase [91-259]. The first phase is marked by one entrance of the instructor [54-90] into the conversation and the second phase by two [201-214, 238-259]. In the final phase [260-298], Lisa and Patrick write up their individual condensed proofs. Similar to Excerpt C, pockets of trivial discussions interfuse the overall discourse, as the students, paired together for the first time, most likely get to know each other. Nevertheless, unlike Excerpt C (and Excerpt B which did not contain as much off-task discussion), this conversation comes to mathematical closure, with each student writing a condensed proof based on an acceptable proof analysis. It would be remiss not to speculate on the effects of the instructor intervention and perhaps even the modified handout on this outcome. The forthcoming analysis takes up these subjects. We begin, though, with a discussion of each of the three phases.

Phase 1: Organization

The conversation opens with Patrick helping Lisa, who is working with an older version of the Solow text, to locate Proposition 1 which Problem 3.18 directs them to use to complete the proof. Patrick recalls "already proving this (presumably Proposition 1)"[6]. In fact, the proof of Proposition 1 appears as an example in the Solow text. Later utterances [26,32, 34] by Patrick, indicate that he actually recalls proving the converse of Proposition 1, although he does not make the critical distinction between the two. Likely aided by the organization of the handout, we see Lisa and Patrick address the key question [5, 9-22] abstract answer [23-30] and mention the applied answer [31] in linear-like progression. While the key question results in some back-and-forth banter, the students settle relatively quickly on "How do we show a triangle is isosceles?" [12]. The reader is reminded here that the students had previously worked with the same proposition in Problem 3.17 and that the key question does not change. The abstract answer, however, is different and consequently generates more discussion. The utterances do not indicate a lack of consensus on the abstract answer centering on Proposition 1 [23-

24]. However, the utterances reveal an awkward, if not incorrect use of Proposition 1. As he writes on his handout (see Abstract Answer in Figure 20), Patrick states slowly, "Proposition 1. Right triangle XYZ has area z squared over four is isosceles" [24]. Rather than restate the conditional (using if-then), Patrick creates a conjunction with the proposition's antecedent and consequent. Lisa questions this move, "So we're saying it *is* isosceles?"[25], but she does not articulate the reason behind her questioning. Other than Lisa's question [25], there is no evidence that points to either of the two students' recognizing an inconsistency between their key question [12] and the abstract answer [24]. Ultimately, the students need to demonstrate that the triangle they are working with is isosceles by showing that it has an area equal to the square of its hypotenuse divided by four. They next, however, begin searching in their folders for the work they completed earlier in the semester on Problem 2.23, the converse of Proposition 1.

Soon after, I enter the conversation unaware that the pair has been interchanging Proposition 1 with the proposition from Problem 2.23. I assume that they are speaking of Proposition 1 when they talk of having "already proved it" [63, 65]. Thus, rather than clarify the difference between a conditional and its converse, my responses [64, 66] likely serve to perpetuate what turns into a somewhat distorted use of the converse throughout their work [126,127]. I do, though, recognize from their utterances [particularly 61], that while they have identified Proposition 1 for their applied answer, they have not necessarily honed their thinking about what it says nor how it will be useful [68]. Thus, I attempt to clarify their abstract and applied answers [68-90]. However, the remaining analysis will reveal that the focus of my assistance may have been misplaced.

Phase 2: Proof Analysis

Working together again, the students accurately identify the hypothesis (A) [91-96] and the conclusion (B) [110,117]. They concur to begin their proof analysis with the fact that $\sin(U) = \sqrt{\frac{u}{2v}}$. Despite a revision of their abstract/applied answers and having together correctly produced the key question, hypothesis, and conclusion, the next series of utterances (see Figure 22) indicates that they are continuing to use Proposition 1 as if they already have an isosceles triangle.

126	Lisa	Wait, . . . okay so, what did we say? Right triangle is one-half base times height. That's not a right triangle. Here we go. Well that the , mmm, I don't, I don't know. Should we start with that? Cause could we start with area and put the w squared over 4? And then cause that's the area of isosceles, but then we could do something with the area of a right triangle. (pause)
127	Patrick	So we have the area equals one half base times height equals w squared
128	Lisa	Over 4

Figure 22. Dialogue between Lisa and Patrick.

While there is a remote possibility that the students were working backwards from the conclusion, a tactic that later proves very fruitful, neither their utterances nor their work (see for example line A1 on Patrick's work) indicates this to be the case. What is more likely, based on previous lines in the transcript [24-26, 31-32], is that the students are still using, albeit in a confused way, the converse proposition from Problem 2.23 to state that the area of the triangle is equal to the square of *w* divided by four. If this is indeed the case, in doing so they have made an implicit assumption that the triangle is isosceles. However, we do not see them carry this assumption through the next steps of their work. Patrick substitutes the lengths of the legs in the given triangle in for *base* and *height* in the area formula [171], but never makes the claim (orally or in writing) that the base and height are equal (Note: In an isosceles right triangle base = height). He and Lisa continue to manipulate the area equation arriving at $2uv = u^2 + v^2$ [179-185]. When they reach a pause, Lisa wonders aloud whether introducing the sin equation from the given is a viable option [190] and they decide together, more or less, that it is not [191-198]. When I enter shortly thereafter, I suggest that their area equation is what they need to produce, not start with [206,208,210]. Before continuing, it is interesting to take notice of Patrick's subsequent work. While there are no utterances in the transcript directly associated to it, Patrick continues the line of thinking that he was pursuing to produce three additional lines of algebraic manipulation that lead to $u = v$ (see Figure 20). Patrick's work then, is a happenstance proof analysis of Proposition 1—the proposition he needed to *use* to prove the proposition in Problem 3.18.

I exit the students' conversation [214] and re-enter a short time later [238]. In the interim, they reorient their focus on the given information for the proposition that they are trying to prove. They discuss and eventually set two ratios for sin equal to one another [219-225]. When I arrive back on the scene, I find them stalled. With time running out, I nudge Lisa through some necessary algebraic manipulations (as Patrick follows along) [238-250]. It does not take long for her forward steps to connect with the last in the chain of backward steps that she has listed [251-256] (see B, B_1, and B_2 on Lisa's work in Figure 21). Most student pairs that I had worked with that day had the moment of realization that they were successful only once they had reached the area equation $\frac{1}{2}uv = \frac{w^2}{4}$ (the one most closely resembling $a = \frac{1}{2}bh$). Lisa, however, comes to the recognition, with my intervention, slightly sooner (as soon as "those (u's) cancel out" [254]) (see Line A5 of Lisa's work in Figure 21). It seems likely that Lisa's earlier recognition is due in part to the time that she and Patrick had already spent discussing and manipulating the area equation. Interestingly, while Patrick led most of the manipulations during the period in which they worked on the area equation [127-128, 171-181], he seems hesitant about the steps Lisa performs to complete the analysis, stating "I think you're going to have to help me" [260]. Patrick never does write out a proof analysis reflecting the up-to-date conversation, and as we see in Phase 3 Patrick and Lisa generally write their condensed proofs individually**.**

Phase 3: Writing a Condensed Proof

Although Patrick initially requests some help, it becomes evident through the conversation that he has intentions of "writing something different" [275] than Lisa in his condensed proof. Lisa shares some of her ideas aloud [274,280,287,288] while she is writing; and as she does, Patrick is responsive to her musings [275, 281, 290, 294]. But, throughout the latter portion of Phase 2 [238-298], specifically the portion that was mathematically productive, he becomes a participant-observer, rather than an active participant. The combination of Patrick's lack of an up-to-date proof analysis and his paucity of pro-active object-level utterances during Phase 2 provide little evidence about *his* thinking. Were it not for the students' individually written condensed proofs, the already analyzed data might leave us with a false sense that Lisa has reached a more expert understanding of the proof in Problem 3.18 than Patrick has. However, their condensed proofs, although individually written, indicate that the pair has developed a similar way of thinking (or mis-thinking) about the proof. Moreover, we see strains from their overall discourse, not just the portion of discourse that resulted in Lisa's proof analysis, influencing their condensed proofs. The reader is invited to now read the condensed proofs in Figure 23 to better understand the analysis that follows.

Lisa's Condensed Proof: To reach the conclusion ΔUVW is isosceles it will be shown the area is equal to $\frac{w^2}{4}$. Given the hypothesis and the definition of sin $\sin(U) = \sqrt{\frac{u}{2v}} = \frac{u}{w}$. By using algebra we showed that $w^2 = 2uv$. Also, knowing that the area of a right triangle equals $\frac{1}{2}bh$ we were able to set $\frac{1}{2}uv = \frac{w^2}{4}$. Through manipulation we were able to arrive at $2uv = w^2$.

Patrick's Condensed Proof: Given that the right triangle *UVW* satisfies $\sin(U) = \sqrt{\frac{u}{2v}} = \frac{u}{w}$, we can by algebra manipulate it to show that $2uv = w^2$. Similarly given that the area of a right triangle by Proposition 1 is equal to $\frac{w^2}{4} = \frac{1}{2}uv$, which can by algebra be manipulated to show that $2uv = w^2$. Hence, the condition $\sin(U) = \sqrt{\frac{u}{2v}}$ is met by the condition area=$\frac{w^2}{4}$, and they are equivalent. Q.E.D.

Figure 23. Lisa and Patrick's condensed proofs for Problem 3.18.

We see in Patrick and Lisa's condensed proofs residue of their discourse related to Proposition 1. Specifically, the condensed proofs indicate that the imperfect use of Proposition 1 that surfaced during the discourse [24-32,127-128] has yet to be fully resolved. Patrick's statement in his condensed proof that "given that the area of a right triangle by proposition 1 is equal to $\frac{w^2}{4} = \frac{1}{2}uv$" is blatantly incorrect. Rather, Proposition 1 states, *if* a right triangle has area equal to $\frac{w^2}{4}$,where *w* is the hypotenuse, then it is isosceles. Lisa may harbor a similar mis-thinking, masked in the more vague statement "knowing that the area of a right triangle equals $\frac{1}{2}bh$ we were able to set $\frac{1}{2}uv = \frac{w^2}{4}$."

We can additionally trace the *form* of both Lisa and Patrick's condensed proofs back to their discourse on the area formula [127-128, 171-181]. This discourse later allows Lisa to set up three backwards steps in the proof analysis (B, B_1, and B_2). The moment of realization that the proof analysis is complete comes to Lisa when her forward step, cancelling a *u* to get $2uv = w^2$, matches her last backward step [254,256]. The partly forward and partly backward form of the condensed proofs closely parallels *how* the students arrive at a realization in their discourse and *where* the realization occurs (when $2uv = w^2$). We see that, in their respective condensed proofs, both Lisa and Patrick provide an explanation of how they arrived at $2uv = w^2$ by manipulating the *sin* equation The partly forward and partly backward condensed proof is more than acceptable. It is of note here because other students in the seminar wrote condensed proofs that were strictly forward, ultimately demonstrating that the area of the given triangle was equal to $\frac{w^2}{4}$. It is reasonable to believe that a strictly forward approach to the condensed proof would have been unlikely for Lisa and Patrick though, given their misuse of Proposition 1.

Summary

This excerpt, not unlike the others, provides us with much to consider in the realm of learning mathematical proof in small-group discourse. Moreover, at this point in the chapter, we have the luxury of comparing and contrasting these findings with the previous three. The introduction of this section brought to the attention of the reader two instructional interventions absent from previous excerpts. We will now consider the effects of the handout modifications and my frequent visits with the pair on their discourse (and consequent mathematical learning).

Lisa and Patrick do address each of the components listed on the handout (key question, abstract answer, applied answer, A [hypothesis], B [conclusion], the analysis of the proof, and a condensed proof) in their discourse. This is in contrast to Excerpt C where Patrick and his partner Jacob do not even mention the key question until I intervene. The research design does not allow for drawing a cause and effect relationship between the presence of the prompts on the handout and the utterances that play out in the discourse. What we learn from the discourse analysis, however, is perhaps even more important. Whether it was the phrase on the handout or any combination of reinforcements received by the students throughout the seminar, Lisa and Patrick negotiate "how do we show a triangle is isosceles?" [12] as their key question. They next turn their attention to the abstract answer and decide on using Proposition 1. The presence of a prompt then, written or otherwise, may serve to focus discourse—in the sense that it influences *what* (the mathematical objects) students discuss. However, as we see throughout the remainder of the transcript, a prompt does not necessarily focus the discourse's *mathematical precision*. The utterances reveal a rather muddled use of Proposition 1 as the abstract answer; so much so that Patrick's condensed proof does not even make mention that triangle *UVW* is isosceles. Alone, this finding may not appear noteworthy. But when considered alongside other findings, important questions are raised. Consider, for example Excerpt C. Neither a key question nor an abstract answer provide focus for Jacob and Patrick's discussion until my entrance into their discussion. It was speculated that, had they spent time to articulate the key question and abstract answer, they might have avoided miscommunication over the hypothesis and conclusion. In the present excerpt, we see confusion surrounding the hypothesis and conclusion *despite* the participants attending to the key question and abstract answer.

Let us now take into consideration my frequent intercession in the student discourse in this excerpt. Once again, we make no claims about cause and effect; offering instead conjectures grounded in the data. First, it is possible that the most significant communication breakdown in this conversation occurred not between Lisa and Patrick, but between the students and me [63-66]. What is clear is that as a limited participant in the discussion, I made an inaccurate assumption about the students' understanding of the abstract answer. Although there existed a hint of evidence to the contrary [61], I presumed that their identification of Proposition 1 was sufficient for its proper use, when in all actuality they were using it interchangeably with its converse. Moreover, by omitting "if-then" when writing down Proposition 1, the students' failed to apply the same attention to detail in their language that they do in an arguably less decisive part of the conversation [96-107] later on. Thus, in the four lines of transcript found in Figure 24, the words "it" and "that" take on different meanings for different interlocutors.

63	Lisa	Oh I thought we already proved it so we could assume that it's true.
64	Me	Yea, you can assume it's true, yes.
65	Patrick	Cause we proved it right there, we proved it the second week.
66	Me	Right. Okay, so you can use that to help you prove this other theorem.

Figure 24. Dialogue between Lisa, Patrick, and Researcher/Instructor.

I endorse the students' use of Proposition 1, unaware of the breach in communication. We cannot predict what might have ensued had I recognized what was occurring. What we do know is that the students begin their proof analysis by assuming that the area of the triangle *is* equal to $\frac{w^2}{4}$. As we have seen, this foray actually establishes the groundwork for the students' reliance on the backward process later on to complete the proof analysis, a pinnacle moment in the overall discourse. But we have also seen that the condensed proofs lack evidence of the role that Proposition 1 played in proving the triangle isosceles.

It seems plausible that my interventions made it possible for the pair to bring the proof to some resolution—albeit imperfect. This is in contrast to Excerpts B and C in which the seminar time ended without the respective pairs of students reaching closure for the proofs. We sense in Lisa's utterances a genuine pleasure at having seen the proof analysis through to its end. "I did it!" [256] she exclaims. It is important not to underestimate how this sense of satisfaction might motivate Lisa in future proof tasks. Nonetheless, it is fair to ask how much of the proof the students did and how much of the proof I did for them. After all, I was heavily involved from helping them set up their proof analysis to coaching their algebraic manipulations. It is additionally reasonable to consider that my verbal presence limited others' verbal opportunities. In particular, the reader of the transcript will no doubt notice that the majority of my utterances are in direct exchange with Lisa. What then, went unsaid, (or unheard) because of my participation? This issue is considered further in Chapter 5.

Excerpt E

Overview

Excerpt E is from the second seminar session on proof. Its presentation here is purposeful. While not chronological, the placement affords us opportunities to compare and contrast it to previous excerpts, in addition to highlighting its own unique features.

The entire excerpt, roughly 18 minutes in length, is the discourse of Amanda, Karen, and Patrick (already introduced), related to Problem 3.9 (see Figure 8). The reader is invited to read the transcript (Appendix I) in its entirety at this time. The discourse can be broken into three phases. Of interest here, is the discursive foci of the interlocutors in each phase and how it relates to their written proof analyses and condensed proofs.

Phase 1: Discursive Focus on the Mathematical Object of "Odd"

The conversation opens with Patrick reading the problem aloud. In the first phase of the excerpt [1-13], lasting just over five minutes, the participants use the definition of odd [3,4,7], interpretations of that definition [6, 10], and algebraic maneuvers [9] to recognize, fairly quickly, that n^2 is in fact an odd integer. In line 10, Patrick succinctly recognizes in Amanda's communication (thinking) the structure of an odd number: "And an even number plus one is an odd number." "Yea, exactly," Amanda says in Line 11, "so it's going to be odd." The modified preoccupational analysis (for three persons instead of two), indicates that this mathematically intense period in the conversation is dominated by pro-active object-level utterances. The exclusive focus of the interlocutors' discourse in this phase is on odd as a mathematical object.

Phase 2: Continued Discursive Focus on the Mathematical Object of "Odd" With Modifications

The utterances in Phase 1 indicate that Amanda, Patrick, and presumably Karen since there is no evidence to the contrary, have convinced *themselves* that n^2 is odd. Yet, similar to Sara and Lisa in Excerpt B, there exists a gap between "seeing it" and "showing it." "Like I just don't know how to order it, and what to put over here," Amanda states in Line 13. A period of silence ensues [14]. Unlike Excerpt B, at no point in the initial phase (or actually at any time) of this excerpt did one of the interlocutors initiate discourse specifically related to managing the proof process. That is, the interlocutors did not verbally utter any key questions, abstract, or applied answers.

At some point, however, whether during the first phase or during the period of silence when no utterances were exchanged [14], Patrick identified the hypothesis and conclusion on his paper (see Figure 25), indicating his thinking about the starting and ending point of the proof.

A: $n \div 2 = 1$ (n is odd) $n = 2k + 1$	B: $n^2 \div 2 = 1$ (n^2 is odd) $n^2 = (2l + 1)$

Figure 25. Patrick's hypothesis and conclusion for Problem 3.9.

In terms of the first line of his hypothesis, we of course know that for $n \div 2 = 1$, n must equal two, which is not odd. A similar explanation holds for the first line of the conclusion. The written statements, however, are likely related to Patrick's thinking on the *definition* of odd. In Line 6, he states "odd if and only if number divided by two has a remainder of 1." The second lines of his hypothesis and conclusion conform to the definition provided from the text that an odd integer is the sum of twice an integer and one. It is significant to note that to show the hypothesis n odd and the conclusion n^2 odd, Patrick chose two different variables, k and l, to represent the integers each being doubled and added to one. "How about this," Patrick states, breaking the silence that follows Phase 1 [15]. He continues to explain his idea to his partners. The focal analysis for the second phase of Excerpt E (Lines 15-18) is in Figure 26.

		Pronounced	**Attended**
15	Patrick	How about this? We know that n squared is equal to this expanded business	By "expanded business" he refers to the expanded form of $(2k+1)^2$. On his paper he has written $n^2 = 4k^2 + 2k + 1$. (In actuality the linear term is *4k*.)
16	Girls	Hmm and Yeah	
17	Patrick	We say that the term n squared is equal to this short hand the different 2l times (plus) 1. We can set those two equal, cancel out the ones. And since we know that two even numbers sum to an even number we're done. Since we can just add the one back in and make them both odd.	By "this short hand" he refers to his representation of the conclusion: $n^2 = (2l+1)$ By " set those two equal" he refers to the "expanded business" ($n^2 = 4k^2 + 2k + 1$) and "the shorthand" ($n^2 = (2l+1)$) He has the following written on his paper: $n^2 = 4k^2 + 2k + 1$ Even $4k^2 + 2k + 1 = 2l + 1$ -1 -1 $2(2k^2 + k) + 1 = 2l + 1$ $2k^2 + k = l$
18	Girls	Yea	

Figure 26. Transcript and Focal Analysis for second phase of Excerpt E.

In setting his "expanded business" equal to his "shorthand" and solving, Patrick cleverly reveals the intermediary steps of the proof. He is in effect, working forward from $4k^2 + 2k + 1$ and backward from $n^2 = (2l+1)$ simultaneously. In Line 17, Patrick's utterances suggests that he is convinced the method fulfills his/their goal, although it is not evident exactly what that is, since they did not verbally articulate it in the form of a key question, abstract answer etc. In Line 17, he states: "And since we know that two even numbers sum to an even number we're done. Since we can just add the one back in and make them both odd." Note that the *focus* of this particular utterance remains on the mathematical object of *odd* (he starts with an odd number, subtracts one to get an even number, and then adds one to regain an odd number). In plain speak, the discourse provides ample evidence that Patrick "gets" what it means to be odd. It "makes sense" to his counterparts and is their focus as well (see Figure 27).

22	Karen	Well it makes sense to me because that's the proposition and we just squared it to get that. And then he just subtracted the one and they'd both be even. And then if you added the one back in they'd both be odd.
23	Amanda	Yea.
24	Karen	Cause an even number plus one is odd.

Figure 27. Utterances by Karen and Amanda.

Sara and Lisa [Excerpt B] also "got" what it meant for a number to be odd, but were unable to produce the steps to "show it." In defining n as $2k+1$ and n^2 as $2l+1$ and then solving for l in terms of k, Patrick provides his group with (discursive) tools that Sara and Lisa did not have. To this point in the conversation, however, the focus on *demonstrating* that a *specific object* was produced (namely n^2) *is* odd still seems secondary to the focus on *verification* of the mathematical object *odd.* We see though, in this phase evidence for the *potential* linking of the mathematical object of *odd* with a *specific* mathematical object, n^2. The group has all of the pieces needed to produce a logical and coherent proof analysis.

Phase 3: Discursive Focus on Demonstrating a Mathematical Object is Odd

This excerpt differs from Excerpt B in another substantial way. Whereas, Sara and Lisa worked for the entire time without my intervention, I enter the conversation in Excerpt E shortly after Patrick has explained his idea [20]. When the group asks for my feedback on the idea, I ask them about their key question [25].What is it that they needed to show [29]? Karen acknowledges they did not write a key question down [26], but Patrick quickly states, "How do you show a number is odd?" [30]. This utterance is significant when placed in context with the analysis to this point. As has been pointed out 1) the focal emphasis of the discourse has been on the mathematical object of odd and 2) multiple utterances [6,7,10,17,22,24] indicate the group members do indeed recognize an odd object. A better framed key question, however, would have been "How do you show the *square of a number* is odd?" leading up to an abstract answer of "show that the square of a number equals twice an integer plus one." Finally, when applying the answer to the specific problem, it is necessary to show $n^2 = 2l + 1$. Patrick provided all of the requisites for accomplishing this in Phase 2.

The group members indicate that to them, Patrick's thinking "makes sense." Phase 3 of excerpt E is predominantly characterized by me questioning and clarifying with the students how they will arrange this thinking in a proof analysis that is convincing to *others* [20-71]. Their work is displayed in Figures 29, 30, and 31. We see in Figure 29 that Patrick's written work effectively ends with his original idea. He makes no further written attempt at analysis. His remark in Line 54, "Well I don't (understand what I am doing). Can you explain it to me?" may be an attempt at humor. It may, however, be indicative of him not recognizing it as *necessary* to further organize or not recognizing *how* to further organize his thinking (communication). Amanda's utterances in Line 50, although rife with demonstrative and subjective pronouns, hint at how she will take Patrick's idea and organize it into her eventual proof analysis (see Figure 28).

		Pronounced	Attended
50	Amanda	Your l would be your 2k problem your factoring. Like you [take] this, and this basically saying that he took a two out of here, and then it would be 2k squared. Then instead of putting that in there, he just made another variable for it. Instead of making it more complicated.	Referencing Patrick's work "here" likely referring to the first two terms "it" likely referring to $2k^2 + 2k$

Figure 28. Focal Analysis for Line 50 of Transcript.

In Figure 30, we see that her proof analysis connects the hypothesis to the conclusion in a logical fashion (it does contain an algebraic error in A4). Karen provides the fewest utterances in the discourse and those that she does provide are mainly reactive and non-object level. Although her proof analysis (Figure 31) takes the form of a more traditional proof analysis (steps in left column/reasons in right column), the organization is less fluent than Amanda's. The latter portion of Karen's proof analysis is a chronology of Patrick's thinking.

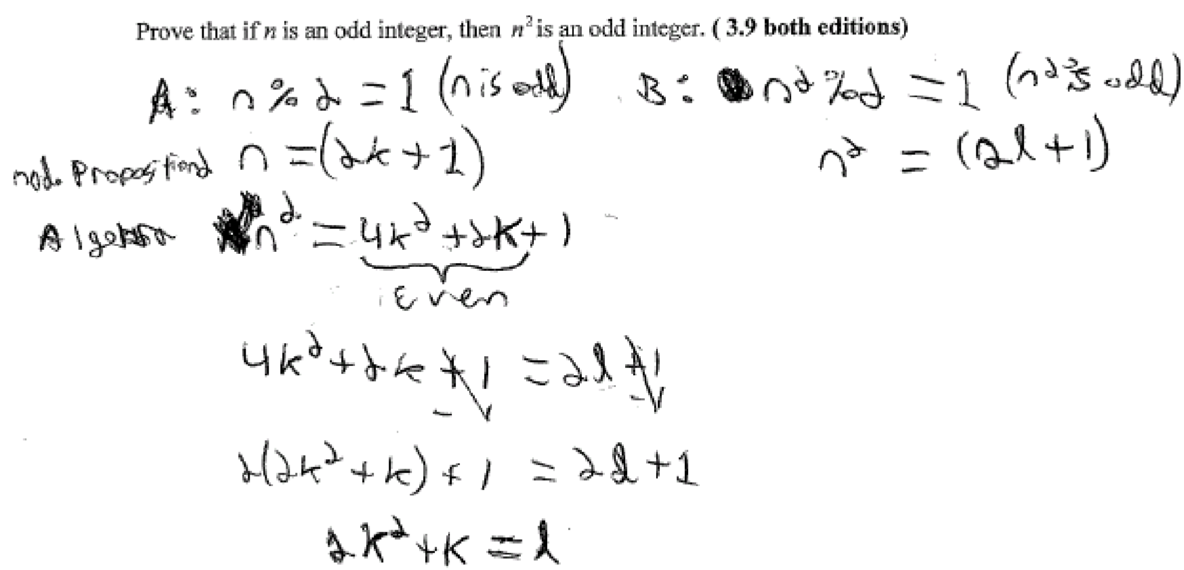
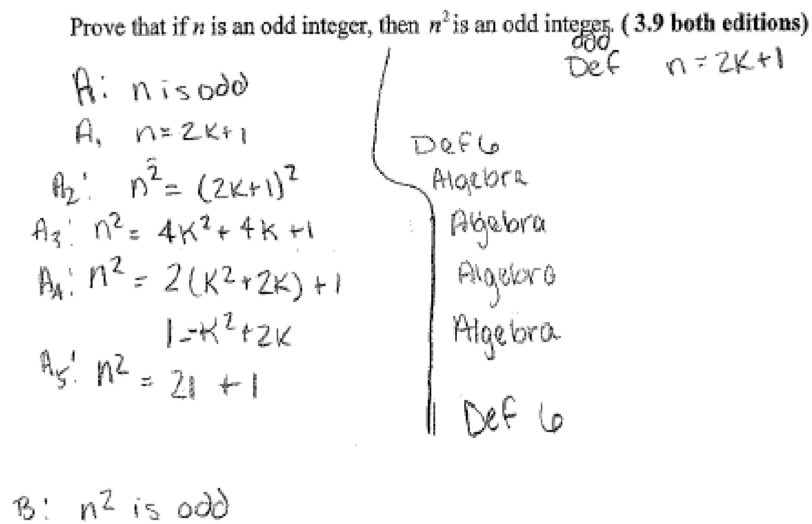


Figure 29. Patrick's work on Problem 3.9.



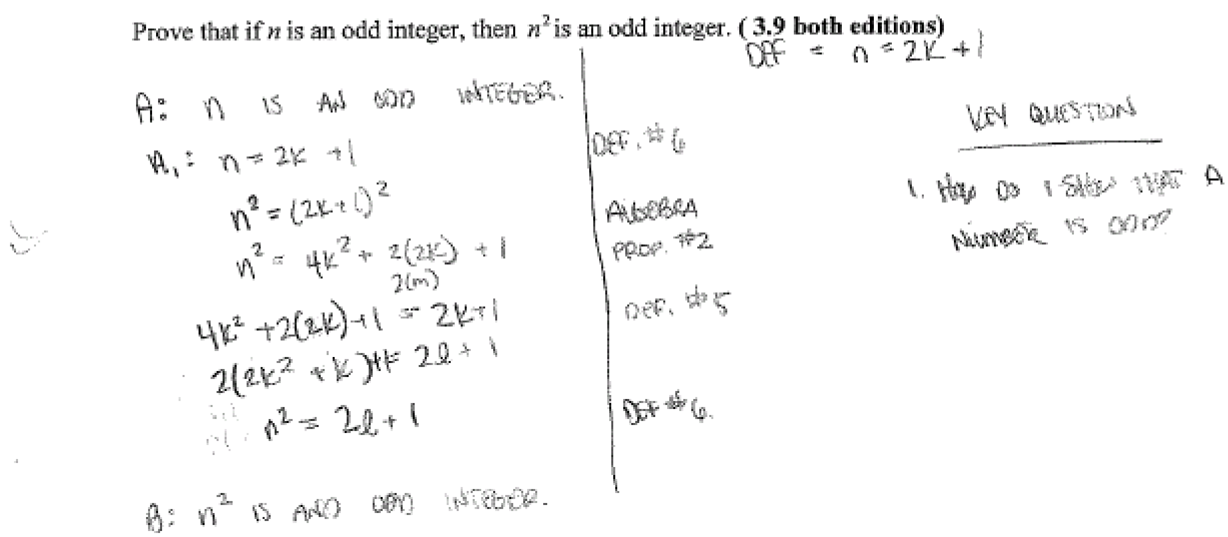
Prove that if n is an odd integer, then n^2 is an odd integer. **(3.9 both editions)**

odd Def $n = 2K+1$

A: n is odd	
A_1 $n = 2K+1$	Def 6
A_2: $n^2 = (2K+1)^2$	Algebra
A_3: $n^2 = 4K^2 + 4K + 1$	Algebra
A_4: $n^2 = 2(K^2+2K)+1$	Algebra
$l = K^2 + 2K$	Algebra
A_5: $n^2 = 2l + 1$	Def 6

B: n^2 is odd

Figure 30. Amanda's proof analysis for Problem 3.9.



Prove that if n is an odd integer, then n^2 is an odd integer. **(3.9 both editions)**

DEF = $n = 2K+1$

A: n IS AN ODD INTEGER.	
A_1: $n = 2K + 1$	DEF. #6
$n^2 = (2K+1)^2$	ALGEBRA
$n^2 = 4K^2 + \underbrace{2(2K)}_{2(m)} + 1$	PROP. #2
$4K^2 + 2(2K) + 1 = 2K+1$	DEF. #5
$2(2K^2 + K) + 1 = 2l + 1$	
$n^2 = 2l + 1$	DEF #6.

B: n^2 IS AND ODD INTEGER.

KEY QUESTION

1. HOW DO I SHOW THAT A NUMBER IS ODD?

Figure 31. Karen's proof analysis for Problem 3.9.

It is noteworthy that the finalization of proof analyses involves no discussion among the group members. After I exit [71], the audio recording reveals the sounds of erasing and then an extended period of silence wherein the participants ostensibly complete their own work. Time was not an issue at this point. Thus, why the participants failed to discuss their continuing work is a point of curiosity. Any attempts to come up with reasons for why this might be would be speculative. Nonetheless, it seems fair to raise the question of whether my presence potentially served to diminish the discourse of the learners. Why did Amanda not share her proof analysis with her partners? How would the group members have organized and presented the thinking of Patrick in a final proof analysis (or would they have), had they been completely on their own? Related questions were raised in Excerpt D. Thus, more will be said in Chapter 5 on this topic.

I do briefly re-visit the group long enough to encourage them to try to write condensed proofs [73]. The participants had little to no practice to this point doing this. For all intents and purposes, their efforts represent first attempts. Thus, the goal here is not to evaluate the quality of what they produced, but rather to examine it in light of their discourse. While the discourse related directly to writing the condensed proofs is thin, and at times off task [75-82], the condensed proofs hearkens back to the foci of the interlocutors' earlier communication. The three condensed proofs are in Figure 32.

Patrick's Condensed Proof
From the hypothesis that the integer n is odd, a state that can be defined as being one more than an even integer, we argue that the quantity n^2 is also odd.

Karen's Condensed Proof
From the hypothesis n is an(d) odd integer, a state that can be determined as being one more than an even integer, we argue that the quantity n^2 is also odd.

Amanda's Condensed Proof
The hypothesis along with the definition of an odd integer yields that $n=2k+1$. Through algebra it can be proven that $n^2 = 2l + 1$, which by definition is also odd.

Figure 32. Patrick, Karen, and Amanda's condensed proofs for Problem 3.9.

Much like his verbal utterances, we notice in Patrick's condensed proof a prominent focus on the mathematical object odd. He writes that odd is "a state that can be defined as being one more than an even integer." He "argues" that n^2 is odd, but the details of *demonstrating* that it is odd are noticeably absent. Karen's proof is a replication of Patrick's—she indicates as much in the discourse [81]. We have seen, starting with her utterances in Line 50 and then in producing a more or less coherent proof analysis, that Amanda has focused her thinking (communicating) not just on what it means to be odd, but also what it means to show a specific number is odd in an organized fashion. Her condensed proof follows suit. We see a distinct beginning (hypothesis and definition of odd integer), middle (through algebra) and end ($n^2 = 2l+1$). While the particulars are left out, she lets the audience know that if one starts with an odd number, $n=2k+1$, it requires just algebraic manipulation to arrive at the odd number that was desired, $n^2 = 2l+1$.

Summary

Excerpt E has provided one final look at a sample of small discourse on mathematical proof by undergraduate mathematics majors. In this excerpt, we had an opportunity to identify nuanced differences in interlocutors' thinking about mathematical proof through examination of their discursive foci (both verbal and written). The reader is reminded here that thinking is taken as a special case of communication, and that discourse is a specialized communication. While the discursive foci of all three interlocutors included object-level rules (focus on the mathematical object odd), there was indication that Amanda was the only interlocutor whose discourse eventually included a meta-level focus (focus on *demonstrating* a specific mathematical object odd).

It is possible that Amanda, a sophomore who had taken Discrete Mathematics the semester prior, may have had more opportunities to explore pertinent meta-level rules of mathematical discourse than her partners, both freshman. The preoccupational analysis of this excerpt, however, reveals limited interpersonal dialogue at the meta-level. As such, there is no evidence to suggest that Amanda's thinking about the meta-level rules advanced the thinking of her partners. The overall conversation then, seems to be similar to what Kieran (2002) calls a "non-mutually productive" grouping—that is, not all interlocutors were similarly successful following their collaboration.

In this excerpt, we see as we have before, that a major hurdle for the learners of mathematical proof is "understanding" but not knowing how to "show." In commognitive terms, it seems as though this conundrum can be interpreted in terms of discursive rules. To understand is to employ the object-level rules of the discourse or to be able to match the object with the word. In this excerpt, all of the learners were presumably able to match the algebraic expressions with the mathematical object (noun) of odd. But to be fluent in a discourse, more is required. One must be able to form sentences, to connect the nouns together in a manner that is coherent to others. In other words, one must be fluent in the discourse's meta-level rules. To write a convincing proof requires the discursant to manage the process of proof. In this excerpt, the discursive foci of Patrick and Karen did not seem to progress to the meta-level needed to produce a proof entirely convincing to others; this despite discursive foci on all of the requisite mathematical objects. It is in essence, the classic learning paradox problem. The paradox is discussed in Chapter 5, as part of the effort to synthesize the data findings. But first, the results from interviews with participants and the results of the *Classroom Community Scale* are presented.

Interview Data Analysis

In this section, I report the findings from the typological analysis of the interview data. The two grounding typologies emanated from the study's research questions. Thus, the findings revolve around the participants' perspectives on two relationships. This section first discusses the relationship of participation in small-group discourse to *learning* mathematical proof. A discussion on the relationship between participation in the seminar and the sense of mathematical learning *community* follows.

Relationship of Participation in Small-Group Discourse to Learning Mathematical Proof

The interview portion of the study reveals three themes relating the study subjects' participation in small group discourse of mathematical proof to their learning. First, the participants valued the exposure to the diverse ways of thinking of their peers as advantageous to their own learning. Second, subjects found participation in small group discourse of mathematical proof an emotionally comfortable way to learn. The third theme encompasses the practicalities involved in the interlocutors' management of a paired discussion surrounding proof.

Appreciation of Diverse Ways of Thinking

The opportunity to work on mathematical proof in small discussion groups that changed each week provided the study participants with exposure to their peers' diverse ways of thinking. The participants were appreciative of this opportunity on a variety of levels. Sara, for example, expressed an aesthetic-like enjoyment of being able to work with others because of her fascination of “how other people think.” In particular, she found that:

> It was interesting to work with the sophomores, because they had more insight. Or even just someone who thought differently. For instance, Patrick has a very interesting way of thinking about things. It’s very interesting to see how he looks at something, I don’t look at it the same. So, that was cool to see.

Karen similarly felt she was a beneficiary of other’s skill sets.

> I think it [working in small groups] was positive. Because every week we got a new group. So you didn’t get to learn with just, say, if we were in a group, I wouldn’t just learn with your math skills. I would learn with Jacob’s math skills or Patrick or Sara. . .And they’re all different. And they all give me new opportunities to look at something. So it did help me after a while.

From the interview data also emerges the sentiment that the discussion proved valuable in seeing, in Lisa’s words, “there’s not just one way to do it.” Tracy similarly commented, “If you had your own way of doing something, I liked how I learned how someone else arrived at coming to a conclusion.”

Hearing a partner’s ideas additionally serves as a mechanism for clarifying and refining one’s own ideas. Nicole explained:

> If I didn’t understand, like I kind of had an idea, but it wasn’t real clear, then I’ll say “I don’t understand” and I’ll have them explain it to me. Everybody is a little different and then I can understand what the technique or problem was.

Similarly, another person’s ideas could help Tracy to complete her own.

> If you have this one conclusion, but you have to have other little steps going up to it, if you don't know those other little steps, your other person's idea might help you out with building those steps to get your conclusion.

Perhaps most exciting is the generative power of discussion in learning mathematical proof that the participants alluded to.

> And like once they would suggest something, it would spark maybe an idea for me to maybe look at it different using the math that I know. And then explain to them, and then just go back and forth. Many different ideas. (Jacob)

> I was partners with Patrick one day. And we had to work on this problem. And I had no idea how to start it. And he said one thing and it triggered something in my head. And I was trying to give him more feedback to his problem. So, it wasn't that I came up with it myself, but he said one thing, I added to his, and we just grew on it. And I like that a lot. (Karen)

Comfort Level in Classroom

The participants viewed the informal conditions of the seminar as comfortable for learning in two main ways. First, some of the students appreciated the non-lecture emphasis of the seminar format because it was congruent with their preference to take an active role in their own learning. Karen, for example commented:

> I come here and I don't know anything about proof or anything like that. But having me sit down there with two or three people in the group and you just give us a problem and you give us the basic of it and we have to figure it out and we have to apply it for ourselves. I like that more. I think I learn more like that.

Jacob recommended continuing with the small-group format because “everybody gets engaged then and everybody has to not just rely on a couple people from the class for input.” Lisa also discussed her desire to try problems on her own, but her comment simultaneously reveals the importance of the safety net that a partner in a small group provides.

> Seeing it on the board and then actually doing it myself helps me a lot. So, I don’t like the lecture thing. So, what you did [small-group format] actually helped me a lot. I like getting in groups, cause then it’s not just you. If you can’t figure it out, hopefully whoever you’re working with can help you. So I like that.

Second, participants found the seminar format comfortable and relatively anxiety-free. Presumably, this allowed them to contribute freely to discussions. Jacob, for instance, liked that the seminar “was informal” and that “you could speak your mind and provide feedback and input based on everybody in this class.” Tracy remarked on the lack of pressure that she felt. “No matter what you said. . . it was either wrong or right, it didn’t matter.” Sara found comfort in the non-competitive atmosphere of the seminar.

> I really did just like how it was informal. It wasn’t something that you needed to go in there and worry about and be like “do I have to compete against other people in smarts to make sure that I’m on the same level with everyone.” I can come in and be relaxed with how things are going.

It is important to acknowledge here that, in addition to the small-group format, the not-for-credit nature of the seminar likely played a significant role in the participants’ comfort level. Students are apt to feel more at ease participating in a non-graded situation, than in a graded one.

Practicalities

The interviews uncovered a variety of perspectives on the practicalities of engaging in paired-discussions on mathematical proof. Overwhelmingly, participants spoke of the practical advantages of working on mathematical proof with a partner. They also commented on the ways in which they navigated hurdles in the discussions to make them more productive. Finally, they indicated that the practical advantages of working with a partner outweighed the inconveniences, which were for the most part surmountable.

Practical advantages. Tracy summarized the practical advantages of working with a partner in this way: "I liked how another person could help you out if you were struggling with something." There was, however, a range of specific ways in which the partner could be helpful. A positive effect of working with a partner, for example, was increased efficiency. "I think what was helpful," Sara said, "[is] if I had one kind of thought and [he/she] said you're going nowhere. [Be]cause I would stop before I spent too much time on it." A partner was also useful for reinforcing right thinking and redirecting wrong thinking. According to Nicole:

> And as far as it was good working together. . . it just helps. It just reinforces what is wrong or right about your thinking. . . Like, if your thinking was wrong originally, and they know for sure. . .they help you realize how to change it so that you're on the right track in your process.

Finally, a partner is valuable not just for feedback on ideas, but also as a possible source of ideas. As Jacob explained :

> If I couldn't start it or he couldn't start it, [there was] always the possibility that one of us was going to be able to start it off at least. Maybe not finish it. Maybe the other person can jump in and work to the end.

Means of navigating discourse. Working with a partner on proofs was not without obstacles. Participation in paired-discourse on mathematical proof entails, as we are about to see, making sense of a dizzying array of ideas—one's own, one's partner's, and the combination thereof. The participants explained how they handled the challenges that arose. Jacob found, for example, that hearing different perspectives from his partner could be "both positive and negative." While a diverse perspective was useful in sparking many different ideas for Jacob and his partner, those ideas might "just go back and forth," becoming difficult to track. Consequently, he would be "sitting there and like 'wait, was I supposed to use that'?" Jacob continued, "A lot of ideas going around, not always documented on paper, [are] hard to do sometimes." Nonetheless, he viewed the situation as unavoidable. "I think that's just part of math, many different problems." As a result, he found that he would sometimes just start going "off on my on my own tangent, on my means of solving it. And then, if I ever got to the end, I would pull my partner and be like, 'here's what I worked out, what do you think?'"

Similar to Jacob, Karen found it hard to understand ideas expressed verbally. A self-described visual learner, Karen explained how she overcame this discussion-based hurdle in one session:

> When people write it down I'll be able to see it. But when people just say it, I have no idea what's going on. So if Patrick would say something to me, I'd ask

> him to write it down and I'd be okay. So I just grow from his, like on that specific problem. I knew that he started it and I added to it and it was fine.

Nicole experienced a somewhat different frustration. Her partner's written ideas were, at times, insufficient for aiding her understanding. Thus, she sought an additional verbal explanation.

> Sometimes somebody that you're working with maybe knows exactly what is going on. And so they don't explain. Or they just write it down really quick and they're like "well here it is." And so that's not helpful when you're working. And if you ask, sometimes they're like, "well, it's just this or this." And you're like "no, I need more explanation" if you didn't understand.

At times interlocutors found themselves negotiating an impasse in the discussion. Although the negotiations retained the air of civility, participants still noted the challenges. Sara reported that the only time discussions were confusing was "if both of us was [were] pretty sure about what we were doing and then we had to sit there and figure out who was right, cause one of them wasn't working." To resolve the issue someone would:

> . . .just stop and like really concentrate on the other person and be like, "okay, you're going nowhere." Or "you screwed up back here" and that's why you think the rest of it's right, but in reality it's not really working like it's supposed to.

Tracy similarly felt a negative of the small groups was when a partner "accepted your idea, but they kind of said that they liked their own idea a lot better." She explained further, how the conversation could be disconcerting.

> For example, if we were trying to come up with the A and B part of proof. I would say it was one way and another person said it would be something different. But it wouldn't be different in a big way, just slightly different.

Relationship of Small-Group Discourse on Mathematical Proof and the Sense of a Mathematical Learning Community

The interview portion of the study provides insight into the participants' understandings of the relationship between learning mathematical proof through small group discourse in the seminar and a mathematical learning community. The predominating theme that emerged was one of communication—participants viewed the seminar experience as a vehicle for learning how to communicate within a specialized community. To appreciate more fully this association, we begin by exploring first, what the participants' understanding of a mathematical community was; and second, the value that they placed on learning about mathematical proof in the seminar.

Participants' Understanding of a Mathematical Learning Community

The participants in the study overwhelmingly identified the provision of support as an important function of a mathematical learning community. Patrick stated it succinctly, "Math is an extremely complex subject and it requires a strong base of peers in order to excel." Embedded in the justifications for the needed support are hints of how participants distinguish themselves from those outside the community. Nicole for example explained the kinship she felt with other mathematics majors:

> We kind of understand each other more. Like our thinking's along the same lines, where sometimes other majors aren't. And they understand what you're going through. Like the math classes are harder than, if you're like accounting.

Similarly, Jacob commented:

> Math is universal, but not everybody can think like a math major. And being in a math community of math majors would allow for a conversation on a math type level, using math terms, and just provide for more insight into just the math world.

Both Lisa and Nicole noticed that the number of math majors on the campus was small and thus felt it was important to know the students who *were* math majors on campus. Lisa said "Knowing who the other math majors are helps, so like if I need help with something I can always go and ask them. Like I'll know who to ask."

The participants had various descriptions for what they thought an undergraduate mathematical community should look like. Lisa's vision was concrete: "I just think of a bunch of kids sitting together at a table and doing math. Doing problems, working on their problems, just getting together and discussing math." Patrick's was similar: "Math majors hanging around white boards with drinks." Other descriptions tended to de-emphasize the math and emphasize characteristics that could contribute to the supportive aspect of community. Tracy, for example, thought teamwork was an important part of an undergraduate mathematical community. Sara felt that a mathematical community should be informal to minimize a competitive feel.

> Rather than being something very structured, and being too structured, sometimes people feel like they're behind. So in an informal setting it's easier for everyone to be caught up and not feel inferior to everyone else.

For Jacob, a mathematical community is a place in which everyone has an opportunity for input, or "free range to speak their mind." Nonetheless, Jacob described a safe and non-chaotic learning environment. "There's got to be methods. Not everyone can shout out at once. It's got to be under control for lack of a better term." Karen described a mathematical community as a classroom where *everyone* felt welcome.

Value of Learning Mathematical Proof

The participants tended to view mathematical proof as foundational to their learning of mathematics. For some, the value of gaining this foundation was pragmatic. Both Patrick and Nicole identified mathematical proof as inherently difficult and felt that they would benefit from the seminar in future coursework. "Proof is a complex topic. . . and having a strong base in it [proof] early on will likely be extremely helpful in my later classes," said Patrick. Nicole summarized it this way:

> For me, it's one of the most hardest concepts. Once you can do it you feel more confident. But I do think that most people struggle with that versus maybe some of the other topics. . .and the more you move up the more difficult they [proofs] become.

Other participants focused on how understanding mathematical proof is integral to understanding the whole of mathematics. Jacob believed that "Mathematical proofs make up the basis of understanding and explaining math." He consequently appreciated the opportunity to learn proofs in a "thorough way." Sara thought that proof "shows you how to back up your ideas more clearly," something she thought was very important since, "in math, you can't get anywhere if you can't show how you got there." Karen viewed proof as permeating all of mathematics—yet in obscure ways. She thought that her future role as a mathematics educator demanded an awareness. In her words:

> Well, I think proof pretty much revolves around all of math. Like every set of problems or sections we do in a book have to revolve around one proof. . .it kind of works in the background…nobody really knows it. It's just like this is what you're supposed to do but you don't know that there was a proof that made it this way. And I think me trying to be a math teacher, I should know that.

Communication as the Relationship Between Small-Group Discourse on Mathematical Proof and the Sense of a Mathematical Learning Community

We saw the emergence of "mathematical learning community as source of support" and "proof as foundational to learning mathematics" as major themes when the participants addressed these topics singularly. What relationship, if any, did the participants draw between learning mathematical proof in the seminar format and being part of a mathematical learning community? The connections drawn by the participants centered on communication. Both Sara and Jacob saw an expression of ideas as the bridge between the two. Sara said:

> I think it preps you for it. Because you are able to learn how to express your ideas and how to accept others' ideas and be able to do that on a larger scale later.

Jacob commented:

> Well, math is universal, a universal language. So if you're able to take your ideas and condense them into a proof, in theory. . .everybody in a math community

> should be able to read and understand that proof and know where you're going with your work.

Embedded in the relationship that Patrick saw between the seminar and a mathematical community is the exchange of ideas or "spread of knowledge." He said, "Well I think ideally a mathematical community would be small collections of informal sessions like the seminar that could transition to a more formal setting to spread knowledge around. Kind of a network." Nicole, Lisa, and Karen, similar to Sara and Jacob, alluded to the seminar as an entryway to a specialized conversation. For Karen, the connection was a realization that "[she] need[ed] to know how to speak correct" not only in her mathematical words, but also her English words when discussing math. Lisa explained that knowing about proof would allow her to feel "more comfortable adding in to the conversation and talking about it." Recognizing that "it (presumably proof) is different than everyday language," Nicole felt that "you maybe understand like where something came from as the other people would. Like others that know a little more or at your same level and you can talk about it or examine it."

Classroom Community Scale Survey Data Analysis

At the seminar's end, the participants in the study completed the *Classroom Community Scale (CCS)*, a survey instrument for measuring community in a learning environment. The reader can find the survey and its scoring key in Appendix A. The *CCS* contains 20 items with half related to connectedness and half related to learning. Participants selected their response to each item from a five-point Likert-type scale: strongly agree, agree, neutral, disagree, and strongly disagree. Each item receives a score from zero to 4, with 4 being most favorable for classroom community. Consequently, scores range from zero to 80 on the overall scale and from zero to 40 on each of the two subscales. Higher scores represent a stronger sense of classroom community. The minimum, maximum, and mean raw scores for eight participants on the overall *CCS* scale and its two subscales are reported in Table 5. (One participant's survey was necessarily eliminated from data analysis due to indeterminable markings.) We see very similar mean ratings by the participants on the learning and connectedness subscales. Each is described in detail next.

Table 5

Descriptive Statistics on CCS Raw Scores (n=8)

	Min	*Max*	*Mean*
CCS Subscale Scores (0-40)			
Learning	24.000	39.000	30.375
Connectedness	20.000	38.000	31.375
CCS Overall Score (0-80)			
Classroom Community	47.000	77.000	61.750

Table 6 provides the minimum, maximum and mean scores of the participants on each of the ten items related to learning. In particular, the items are concerned with "the use of interaction within the community to construct understanding and the extent to which learning goals are satisfied within the classroom setting" (Rovai, 2002, p. 202). Item 20 resulted in the highest mean (M=3.625) closely trailed by Item 4 and Item 18 (M=3.5). All three items were reverse scored, meaning the higher ratings reflect a stronger sense of community. As such, we see that all participants disagreed, some strongly, that the seminar did not promote a desire to learn, that it was hard to get help when they had a question, and that their educational needs went unmet in the seminar. However, a mean rating of 2.000 on Item 12 indicates that participants did not tend to disagree that the seminar resulted in modest learning. This result might be attributed to the once-a-week format and non-credit nature of the seminar. Even so, there was a tendency of the group toward agreeing that there were ample opportunities to learn in seminar (Item 16, M= 3.250). We also see in Item 8 (M=2.125) that participants were not particularly comfortable exposing gaps in their understanding in seminar.

Table 6

Descriptive Statistics on CCS Learning Subscale Items (n=8)

	Items	*Min*	*Max*	*Mean*
2.	I feel that I am encouraged to ask questions,	2.000	4.000	3.250
4.	I feel that it is hard to get help when I have a question.	3.000	4.000	3.500
6.	I feel that I receive timely feedback.	1.000	4.000	3.250
8.	I feel uneasy exposing gaps in my understanding.	0.000	4.000	2.125
10.	I feel reluctant to speak openly	2.000	4.000	3.000
12.	I feel that this course results in only modest learning.	1.000	3.000	2.000
14.	I feel that other students do not help me learn.	0.000	4.000	2.875
16.	I feel that I am given ample opportunities to learn.	2.000	4.000	3.250
18.	I feel that my educational needs are not being met.	3.000	4.000	3.500
20.	I feel that this course does not promote a desire to learn.	3.000	4.000	3.625

Table 7 shows descriptive statistics for items related to connectedness. Once again, higher ratings should be interpreted as a stronger sense of classroom community, with the value of 4 as the highest possible overall rating. We see in Item 1 and Item 19 that participants tended to feel, some strongly, that their classmates in seminar were both caring and supportive. Furthermore, on average participants did not to feel isolated in seminar (Item 9, M=3.375) and they were trusting of each other (Item 11, M =3.375). Participants generally felt that they could rely on others in the seminar (Item 13, M = 3.125). However, they did not view others as being highly dependent on themselves (Item 15, M=2.250). That is, participants did not believe that their own contributions and participation were especially helpful to their classmates.

Table 7

Descriptive Statistics on CCS Connectedness Subscale Items (n=8)

	Items	*Min*	*Max*	*Mean*
1.	I feel that students in this course care about each other.	3.000	4.000	3.500
3.	I feel connected to others in this course.	1.000	4.000	3.125
5.	I do not feel a spirit of community.	1.000	4.000	3.000
7.	I feel that this course is like a family.	2.000	4.000	3.000
9.	I feel isolated in this course.	3.000	4.000	3.375
11.	I trust others in this course.	2.000	4.000	3.375
13.	I feel that I can rely on others in this course.	1.000	4.000	3.125
15.	I feel that members of this course depend on me.	1.000	3.000	2.250
17.	I feel uncertain about others in this course.	2.000	4.000	3.125
19.	I feel confident that others will support me.	2.000	4.000	3.500

Summary

This chapter has presented the findings of the study in three main sections. The first section, the study's hallmark, provided a high-resolution analysis for each of five excerpts of small-group discourse on mathematical proof by undergraduate math majors. In the broadest sense, the analyses probed the effectiveness of communication and resulting mathematical productivity of the discourses. However, the analyses also provided microscopic looks at the various intricacies of learning mathematical proof. The second section summarized the results from the interview data. Specifically, findings were organized as they related to *learning* mathematical proof through small-group discourse and to participants' sense of *community.* Lastly, results from the *Classroom Community Scale* survey were summarized. The next and final chapter includes a comprehensive synthesis of the findings as they relate to the study's research questions. In addition, recommendations for future research are given and implications for the teaching and learning of mathematical proof are discussed.

CHAPTER 5

DISCUSSION

Mathematical proof has traditionally been difficult to both teach and learn. Chapter 2 considered the multi-faceted nature of this predicament in detail. Two broad generalizations bear repeating here. First, lecture methods, often used in the college mathematics classroom, may not fully reflect the way that *mathematicians* do mathematics. It is not surprising then to see emerging efforts to encourage greater student participation in academic discourse (e.g., King, 2001). Second, where purely cognitive and social constructist (not to mention behaviorist) research frameworks have come short in illuminating answers to nagging questions related to the learning of mathematics, a commognitive theoretical framework holds promise. Specifically, commognition takes thinking as a specialized case of communication (Sfard, 2008). Using a commognitive approach, this study has attempted to uncover the nature of learning mathematical proof by freshman and sophomore math majors in a seminar utilizing a small-group discourse format. The two main research questions of the study were:

1. What is the nature of the relationship of undergraduate mathematics majors' discourse of mathematical proof to their learning of mathematical proof?
2. What is the nature of undergraduate mathematics majors' sense of community in a seminar utilizing a small-group discourse format for the learning of mathematical proof?

In this final chapter, after a brief summary of the study, I present a synthesis of the research findings related to each of the research questions. This is followed by a discussion on related future research. A section on the implications of the study for teaching precedes the final remarks.

Summary of Study

This study examined the learning of mathematical proof of undergraduate mathematics majors, primarily through the lens of discourse. The setting for the study was a zero-credit seminar for freshman and sophomore mathematics majors that convened during fall 2008 at a small private Catholic university in Western Pennsylvania. Seventeen audio recording samples of small-group discourse on mathematical proof were collected over a period of six seminar sessions. Additional data collected included student work, pre- and post-seminar written responses, demographic data, pre- and post-mathematical proof survey, and the *Classroom Community Scale* survey. During the first week of the spring 2009 semester, interviews with participants addressed their perspectives on learning mathematical proof and on their sense of learning community in relation to the seminar.

Data analysis included the transcription of all samples of small-group discourse and all interviews. After close reading of the entire data set, five samples of small-group discourse were chosen to undergo high-resolution analysis using the specially developed tools of focal and preoccupational analysis (Sfard and Kieran, 2001). Focal analysis and preoccupational analysis are concerned with the mathematical content of and interlocutors' engagement in the conversation respectively. Taken together the two analyses provide the researcher insights into the effectiveness of the interlocutors' communication (thinking). Typological analysis was used to summarize the interview data and simple descriptive statistics were calculated for the *Classroom Community Scale.*

Overall, the findings were presented qualitatively, using thick description. Each of the five excerpts of small-group discourse could stand alone in its own right—offering the reader microscopic looks at the learning of mathematical proof. In this chapter, I attempt to begin to spin together the complexities of learning mathematical proof by synthesizing the findings from the discourse analyses, interview analysis, and the *Classroom Community Survey.*

Synthesis of Findings

Arguments concerning the paradox of learning trace as far back as Plato. Some things, it seems, are "unlearnable because they must be known before the process of learning [can begin]" (Honderich, 1995, p. 476). So it is with learning mathematics. One must become skilled in discourse that is mathematical. Sfard (2008) states, "Some familiarity with the objects of the discourse seems a precondition for participation, but at the same time participation in the discourse is a precondition for gaining this familiarity" (p. 161). This seems especially true for learning mathematical proof—a particularly vexing topic for learners. Earlier chapters discussed the criticism of traditional lecture-based courses, which have a limiting effect on students' participation in discourse. We also noted recent movements towards the use of more dialogical discourse in the mathematics classroom. This study has sought to describe the nature of learning mathematical proof that occurred in a zero-credit seminar featuring small-group discourse. According to Sfard, answering the "question of how mathematists [learners] manage to overcome this inherent circularity of processes of learning and of investigating" is a foremost task for the researcher of mathematical thinking (p.161). In this final chapter, I present a synthesis of this study's findings in relation to the two research questions. These findings aim to initiate a new discussion about the learning of mathematical proof.

Research Question 1

Mathematics is a specialized discourse. To learn mathematics is to become increasingly skilled in its discourse. To aid in the discussion of Research Question 1, I will employ a diagram consisting of two concentric circles (see Figure 33). The innermost circle represents expert mathematical discourse. In other words, it is the well-defined discourse of the mathematical scholarly community. This discourse includes consensually endorsed narratives, routines, objects, and mediators. The outer circle represents the discourse of learners of mathematics in general, and this study's participants in particular. The diagram will be used to discuss the relationship between the expert discourse and learner discourse. Before discussing specifics, though, consider the potential of the diagram for representing learning in general. First, we might view learning (the evolution of skilled participation in expert discourse) as the shrinking of the outer circle's radius, such that over time the learners' discourse more closely approximates that of the experts'. Eventually, the two circles become one (see Figure 34). Alternatively, perhaps the way to overcome the inherent circularity of processes of learning discussed earlier is to "chip away" at the circumference of the inner circle—such that it eventually disintegrates and the two become one (see Figure 35). Learning is a complex human phenomenon and diagrams are mere approximations of the phenomenon.

I am not advocating one diagram or the other. Rather, I suggest the process of learning might be both a shrinking of the outer radius and the disintegration of the inner circumference.

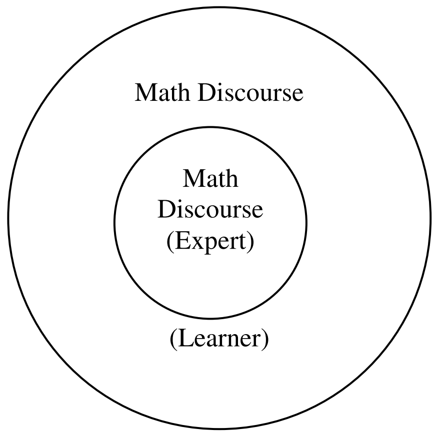


Figure 33. Relation of learners' mathematical discourse to expert mathematical discourse.

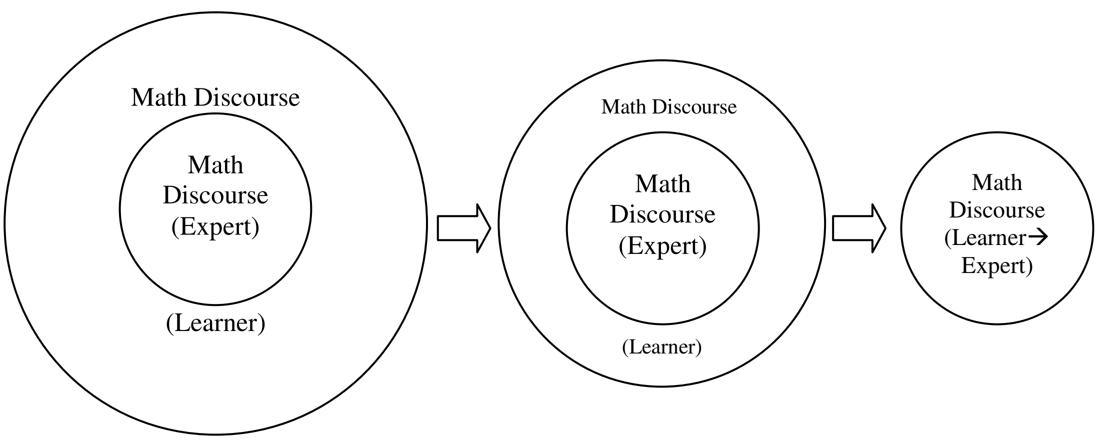


Figure 34. Learning as a shrinking of the radius of learner discourse.

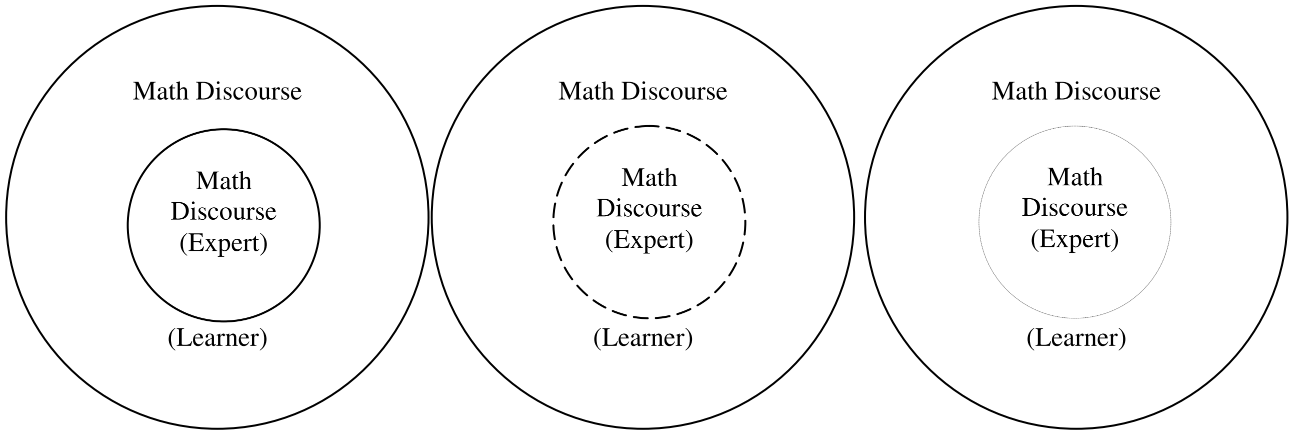


Figure 35. Learning mathematics as disintegration between the boundary of learner and expert discourse.

Learning Environment

The data suggests participants found engaging in discourse with their peers in small groups to be both helpful and comfortable in their beginning efforts to learn mathematical proof. Participants, for example, thought that they "learned best" by being actively involved in what they were learning—something that the small-group discussion format required. They also appreciated having peers to work alongside, valuing the diverse ideas brought to the table and the feedback. Of all items on the *Classroom Community Scale*, Item 20 received the highest mean rating (M = 3.625). The item pertains to the course as it promotes a desire to learn. It would seem, at least from the perspective of the *learner*, that a zero-credit freshman/sophomore seminar utilizing small- group discussion is a favorable way to *approach* learning mathematical proof (see Figure 36). This finding resonates with the results of Springer, Stanne, and Donovan's

(1999) meta-analysis on the effects of small-group learning on undergraduates in STEM courses. In particular, one result of the meta-analysis was that various forms of small-group learning effectively promote more favorable attitudes toward learning.

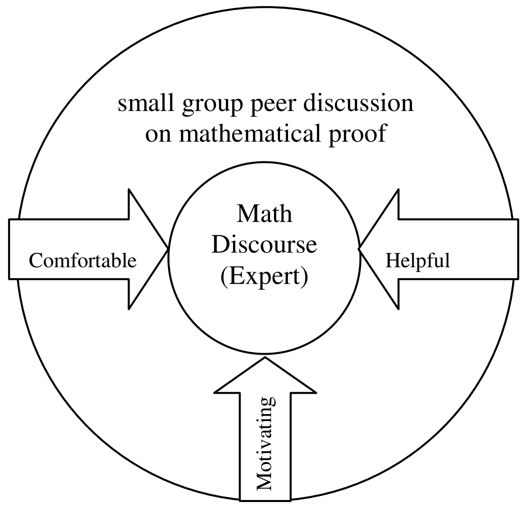


Figure 36. Small-group peer discussion as comfortable, motivating, and helpful to learners in approaching mathematical proof.

Discursive Entry Points

The seminar intended to project a more authentic experience of mathematics. To do this we created a learning situation in which social interaction, imagery, heuristics, and intuition could precede proof construction (Sriraman, 2004). We saw evidence in Excerpts A and B, however, that the paired-discourse of peers may serve to help them organize and clarify thinking on familiar content. But the collaboration itself may do little in the way of helping them to penetrate a more expert discourse. Similarly, we encountered discourses in which peers were stuck in a cycle of "understanding" something but "not able to show it" (Excerpt B, Excerpt E). The analysis of discourse in this study revealed that within small-group discourse on mathematical proof, there may arise *natural* opportunities to steer the discourse in the direction of increasing sophistication—what I have called discursive entry points. This finding potentially gets at the heart of the learning paradox. Paired discourse between peers may provide learners an "un-artificial" way in which to "bump into," so to speak, those things that are unlearnable because they must be known before the process of learning can begin. Of course, we cannot expect the learner to recognize that such an opportunity is occurring since the very thing they are bumping into is unknown to them. It is hypothesized that, with the skilled intervention of an expert in the discourse, these entry points might be "ripe for ripening" (from Vygotsky's *Thought and Language,* as cited in Crain, 2005, p. 240).

Skilled intervention in peer discourse on mathematical proof would involve knowing both *when* and *how* to intervene. This study has started to uncover points in discourse on proof when the learner might benefit from such intervention and *speculated* on how intervention could precede. The analysis found, for example, that potential discursive entry points in discourse on mathematical proof occur:

- *When the interlocutor(s) is aware that an approach to proof has failed* (Excerpt A). Intervention might include asking interlocutors to explain *why* the approach failed in relation to that which they are trying to demonstrate. If a trial-and-error approach is used, interlocutors could be asked to compare the success of various approaches in relation to the proof process.
- *When the interlocutor(s) raises questions specifically related to managing the process of their proof* (Excerpt B). It is likely that the question arose for good reason. Thus, interlocutors should be encouraged to consider these naturally

arising questions and articulate specific answers to them, rather than answer them based on a gut feeling or dismiss them altogether.

Along the same lines, it seems possible that as researchers/educators identify ad hoc routine courses of action in paired-learner discourse, they can determine how to assist the interlocutors in modifying those routines in mathematically productive ways, steering them ever more closely to fluency.

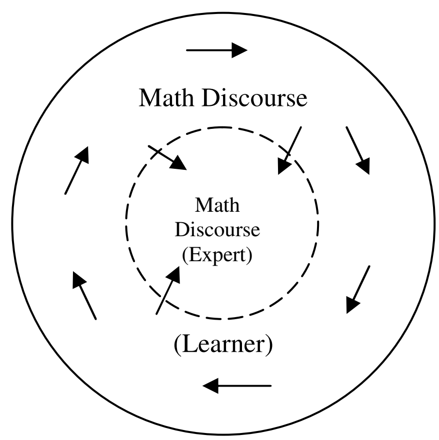


Figure 37. Discursive entry points.

Note: Some discursive utterances (arrows) by interlocutors may be particularly valuable opportunities for moving learners toward a more expert discourse. These are called *discursive entry points* and are represented by arrows with initial points in the outer circle and terminal points in the inner circle.

Factors of Small-Group Discourse Affecting the Learning of Mathematical Proof

This study has revealed several complex factors that influence small-group discourse on mathematical proof. Each of the factors will be discussed individually, as it seems possible that each one may uniquely play a role in how learners can infiltrate a more expert discourse. However, within actual discourse the factors were not isolated. In other words, multiple factors played into a single discourse, with varying levels of prominence and interrelatedness. Accordingly, the diagram in Figure 38 partitions the factors into circle sectors so that we might visualize each factor's potential influence on learning mathematical proof. But dotted lines separate the sectors to represent fluidity/overlap between factors in actual discourse. The ordering of the discussions of each of the factors is a deliberate attempt to illustrate how the factors potentially intersect in actual discourse. I underscore here, though, that this study has only begun to *reveal* the influencing factors of small-group discourse and the learning of mathematical proof. Further study into each factor and the avenue that it might provide into a more expert mathematical discourse is needed. At times, I take the liberty of hypothesizing about this potentiality.

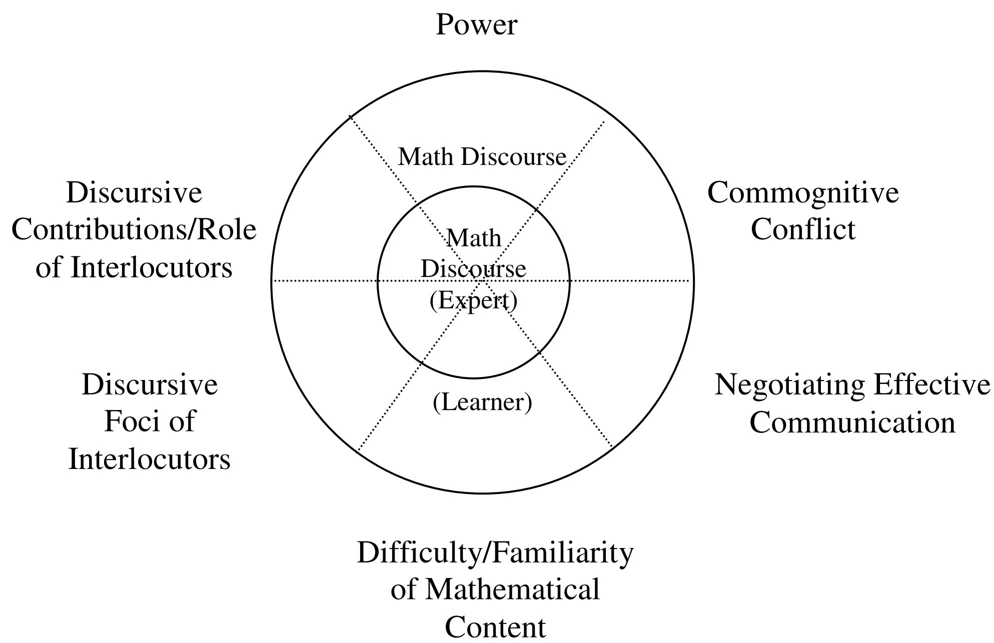


Figure 38. Factors affecting small-group discourse on mathematical proof.

Discursive contributions/roles of interlocutors. One interesting finding of the study relates to the different types of discursive roles that the interlocutors took on in discussions. We saw, in more than one excerpt, that one interlocutor may more or less function as the manager of content (especially in the discourse's opening)—offering utterances on familiar and relevant mathematical objects and algebraic manipulations. Sherri in Excerpt A, Sara in Excerpt B, and mainly Amanda in Phase 1 of Excerpt E all served in this capacity. In a few excerpts, we also saw interlocutors as burgeoning facilitators of the proof process. In Excerpt A, for example, Jacob suggests to Sherri that they try a backward approach to the proof. In Excerpt B, we saw repeated instances of Lisa offering questions that drew attention to tactics. In contrast, in Excerpt C, we saw neither interlocutor assume such a role.

More research on the questions of how learners balance the discussion of content with process and how that balance leads to a more expert discourse is needed. Previously, it was suggested that learners' questions related to process, such as Lisa's in Excerpt B, may serve as natural entry points to steer the discourse toward sophistication. Written reminders similar to the ones provided in the student handout in Excerpt D (the handout included places for students to write the key question, abstract answer, proof analysis and condensed proof) may also be a way to prompt discussion related to process. As we saw in Excerpt D, however, prompts may serve to initiate discussion on process; but they do not ensure its clarity.

Discursive foci of interlocutors. Closely related to the discursive roles of interlocutors, if not completely overlapping, is the discursive foci of interlocutors. When I spoke of the discursive roles/contributions of the interlocutors, I did so in broad sweeps. The interlocutors' discursive roles/contributions are perhaps the magnified versions of their discursive foci. By closely examining discursive foci, such as was done in Excerpt E, we get a nuanced look at the interlocutors' thinking (communicating). The analysis of that excerpt raised the distinct possibility that when learners say they "get" or "understand" a proof, but do not know "how to show" it, their discursive focus (thinking) may be restricted to the mathematical object itself and hence take on a verification-like quality. A discursive focus on demonstrating the mathematical object (meta-level focus), however, is likely more productive. The distinction between the two foci is no doubt incredibly fine. It seemed in Excerpt E that Amanda was able to make the subtle shift between focusing on the mathematical object odd, and focusing on demonstrating the mathematical object odd. It was not evident that her partners fully made this transition. Although, it seems possible that they could have, had Amanda spent time discussing her final proof analysis with them (she did not). A potentially fruitful line of future discourse analysis research would be to seek out and identify any patterns that might exist in instances where there is a noticeable change in the discursive focus of the interlocutor(s), namely from an object-level focus to a meta-level focus.

Difficulty/familiarity of mathematical content. A nagging question that arose from the discourse analysis, especially in Excerpt A, relates to interlocutors' familiarity with mathematical content and the discursive roles/foci that they assume. The Solow text is written in such a way that various proof techniques are introduced mainly using content from algebra, geometry, and trigonometry—all content normally familiar to math majors. This allows beginning learners of proof to concentrate their efforts on the approaches to proof. Is it possible, though, that it could, at times, have an opposite effect? It was hypothesized in Excerpt A, that Sherri's familiarity with right isosceles triangles may have left open the door for her to treat the task as a familiar "problem to be solved," rather than an opportunity to make sense of the logical whole that is proof. Perhaps we see strains of the same in both Excerpt B and Excerpt E where all interlocutors are certain beyond doubt that they have an odd number, yet find themselves at great loss in "showing" it. Does the obviousness of odd loom so large in the foreground that it impedes the focus (exploration/intuition/social interaction) on the *process* of proving? Suppose learners were asked to work on a proof in which the mathematical content was less than familiar—in which they had to work much harder to convince *themselves* of mathematical truth. Would there exist more of a discursive balance in foci—between the object-level and meta-level rules of mathematics? This is yet another intriguing area for future discourse research in the area of mathematical proof.

Negotiating effective communication. Mathematics was famously described by Bertrand Russell "as a subject in which we never know what we are talking about, nor whether what we are saying is true" (from Russell's *Recent works on the principles of mathematics*, as cited in Sfard, 2008, p. 129). Indeed, "what we talk about" in mathematics is inherently abstract. In this study, we have worked from the premise that mathematics is a distinct discourse—set apart from others in the sense that the objects of the discourse are discursive constructs (Sfard, 2008). So how do we ever communicate about mathematical objects? Recall, according to Sfard, that participants in communication rely on mediators, or perceptually accessible objects, that assist the actor in performing the prompting action and the re-actor in being prompted. They are "often artifacts produced specially for the sake of communication" and "can have auditory, visual, or even tactile effects on individuals" (p. 90). In laymen's terms, Jacob perhaps best described the challenge of beginning discourse on mathematical proof that lacks effective communication mediation. He spoke of a situation in which "a lot of ideas go around," but are "not always documented on paper." Ultimately, this makes learning mathematical proof with a partner, in Jacob's words, "hard to do."

We see throughout the interview data and the excerpts of small group discourse, instances of interlocutors' attempts at mediating discourse on proof. In Excerpt B, when Lisa suggests that they try the proof coming in from a different angle, Sara's immediate response is "Show me what you mean." An arrow on Lisa's paper likely serves as one example of an artifact that she used to mediate the situation. Similarly, Karen spoke of the need to see things “written down” to understand them—and of directly asking her partner Patrick to do so. Nicole’s comment, however, points to the complexity of mediation. Seeing her partner’s work was not always enough to help Nicole understand their thinking. She needed an accompanying explanation. This finding suggests that the prompting *action* of mediation may be as crucial as the perceptually accessible objects that serve as mediators. What are the artifacts and the prompting actions that facilitate effective communication related to mathematical proof? A discursive line of inquiry that pursues this question holds enormous potential for contributing to an understanding of what it means to learn mathematical proof.

Commognitive conflict. We have touched on the significance of the role of mediation in making communication effective. What are the circumstances that contribute to ineffective communication? Sfard (2008) describes commognitive conflict as the situation that arises when communication occurs across incommensurable discourses—those that differ in their *use* of words and mediators or in their routines. The analyses in this study suggest that a very basic source of commognitive conflict in interlocutors’ small-group discourse on mathematical proof comes from frequent usage of demonstrative pronouns. To begin with, the objects of discourse on mathematical proof are abstract. Pragmatically speaking, referring to them using demonstrative pronouns muddies the discursive waters. In Excerpt B, we saw that back to back uses of the word “that’s” by Lisa and Sara could have resulted in any combination of meanings. While the women presumably did not find its use problematic in their communication, the analysis suggested strains of commognitive conflict, both interpersonal and intrapersonal.

A slightly different manifestation of commognitive conflict related to demonstrative pronouns in the classroom may relate to the combination of differential in expertise between interlocutors (in this case teacher and student(s)) and of assumptions made by interlocutors. The discussion builds on two premises. First, as the teacher in the seminar for this study, I operated under authentic classroom circumstances. I was constantly circulating between small groups to lend support and assistance. As such, I did not observe first hand any of the student discussions from beginning to end. Second, as the teacher, I was thoroughly prepared and versed in the proofs that the students were working on.

We saw in Lines 63-66 of Excerpt D how the interlocutors' use of the word "it" failed to signify their thinking on Proposition 1. Instead, I assumed that their thinking was commensurate with my own and, in turn, endorsed an imperfect narrative. Unfortunately, it was only after reading and analyzing the transcript, that I realized the ramifications of this interchange. The same blend of circumstances, consisting of a differential in expertise and assumption without clarification, may have had a distinct impact on the level of success of the discourse in Excerpt E as well. We saw, in Line 50 of that excerpt, signs that Amanda was thinking about the overall demonstration of n^2 as an odd object. Line 50, however, is strung together with demonstrative pronouns (this, it, that, here, there). Unless a person's mathematical thinking included not only the mathematical objects to match the pronouns to, but also the meta-level framework, the sentence will make little sense. Here again, I assumed that Amanda's thinking was close to my own and even said as much in Line 57—"And I think Amanda is almost there." But notice, in Figure 39, the next few lines following Amanda's explanation in Line 50.

50	Amanda	Your l would be your 2k problem your factoring. Like you ___ this, and this basically saying that he took a two out of here, and then it would be 2k squared. Then instead of putting that in there, he just made another variable for it. Instead of making it more complicated.
51	Patrick	Just condensing things.
52	Karen	I understand
53	Amanda	I get what he's doing
54	Patrick	Well I don't can you explain it to me?

Figure 39. Dialogue between Amanda, Patrick, and Karen.

Karen said she understands. Recall that Karen's eventual proof analysis/condensed proof have a heavy focus on the outline of Patrick's thinking in Lines 15 and 17, but *not* Amanda's explanation in Line 50. Moreover, Patrick comments in Line 54 that he does not understand what he is doing. Again, it may be an attempt at humor. But he might also not understand Amanda's meta-level use of his object-level thinking. One can only wonder if the learning outcome (as evidenced by the proof analysis and condensed proof) would have been different for Patrick and Karen had I encouraged Amanda to re-communicate her thinking in Line 50 with her peers using more explicit word use and mediators. This section, which has discussed the potential commognitive conflict that can arise when there exists a differential in interlocutors' expertise, hints at the sixth and final factor to be discussed—power.

Power. Sfard (2008) states: "Doubtlessly, therefore, the resulting shape of all the individual discourses involved is a function of power relations among interlocutors" (p. 146). Excerpt C provided an especially noteworthy example of a discourse in which power issues played a pervasive role. We saw a case in which Patrick conceded almost immediate precedence to Jacob's thinking. What is fascinating is that the concession seems to have basis not in a comparison of ideas related to the proof task, but rather on a comparison of prior coursework. This instance suggests how prominent the issue of power might be in small-group discourse of undergraduates when learning mathematical proof. This example may represent a metaphorical sharp edge in moving from classrooms dominated by univocal discourse to those more dialogic in nature. At its core, this is a discussion on sociomathematical norms and what counts as acceptable mathematical explanation (Yackel & Cobb, 1996). In learning situations in which learners are more or less excluded from participating in the discourse, truth will reside with the authoritative voice. In this case, Patrick judged authority in this traditional sense—based on comparison of he and his partner's transcripts. As classrooms shift to a more dialogic nature, reformers hope that the litmus test for what interlocutors count as acceptable will be guided more by the truth of mathematics, and less by social cues. This is not to say that expertise will be devalued. Rather it may open the door for a more authentic mathematical experience that includes social interaction, imagery, heuristics, and intuition (Sriraman, 2004). After all, we saw that despite being a year behind Jacob in the calculus sequence, Patrick had important contributions to make to the productivity of their discussion. It is interesting to note that on the *Connectedness Subscale* of the *Classroom Community Scale*, the item receiving the lowest rating was the statement "I feel that others in this course depend on me." It seems as though participants did not recognize the value of their own contributions to discussions on proof. We saw, in Excerpt C, an additional but interrelated way in which power comes into play. My voice prompted Jacob to revise his hypothesis and conclusion. Patrick's similar comments on the matter did not. Thus, in this excerpt, my intervention seemed beneficial if not completely necessary in this particular learning situation.

Sfard (2008) states that "in the process of mutual discursive attuning, one of the participant discourses would often be privileged over all the others, that is recognized by the interlocutors as the paradigmatic case, which sets the rules for all the interlocutors" (p. 145). In school-learning this is usually the discourse of the teacher/grown-up. Chapter 4 presented five excerpts of small-group learner discourse with varying levels of teacher intervention into the discourse (see Figure 40). Here, let us review the influence of my intervention in each of the excerpts in order to make comparisons and raise conjectures. Before doing so, however, let me state that my only assertion here is that my voice represents that of a privileged participant in the discourse. I make no claims about it being unique, prototypical, or even effective.

Excerpt A	**Excerpt B**	**Excerpt C**	**Excerpt D**	**Excerpt E**
No intervention	No intervention	Moderate Intervention	Frequent Intervention	Prolonged Intervention

Figure 40. Varying levels of teacher intervention into discourse.

Let us begin by looking at the excerpts in which the intervention was greatest. Then we will work our way back to excerpts in which it was non-existent. In both Excerpt D and Excerpt E, we saw that, during my frequent and/or prolonged entrances into the discourse, one or more interlocutors' participation generally decreased. It is not justifiable to claim direct causation between my entrance into a conversation and certain outcomes of the discourse. However, discussing possible correlations seems valuable in shaping future study. Taking Excerpt E as an example, two things were especially disconcerting. First, during my presence in the conversation, there were fewer utterances from Patrick and Karen and more from Amanda. In the end, Amanda's proof bore much more sophistication than her partners' did. Secondly, after my exit from the conversation, the interlocutors fell into relative silence. Rather than further discuss my challenge to them to "convince someone else in a mathematical way" [55], the interlocutors apparently completed their proof analyses individually. One cannot help but wonder if in the minds of students, once the authority speaks, the conversation is over.

Recall that Jacob and Sherri completed their work in Excerpt A with no assistance from me. Similarly, the transcript of Sara and Lisa's collaboration analyzed for Excerpt B, does not contain my voice (although I did discuss the proof with the girls after the seminar ended/audio recorder was turned off). While Excerpt A was characterized by closure and relative success with the proof analysis, the analysis revealed that the actual productivity of the discourse may not have been optimized. The situation was similar to the one just discussed (with Patrick and Jacob). Only in Excerpt A, Jacob's discursive utterances were of potential value to his partner Sherri. But she did not seem to attend to them. Would Sherri have been more receptive to the same ideas had they come from teacher intervention? It was specifically hypothesized in Excerpt A, that the discourse contained particular opportunities—called discursive entry points—where a more skilled or authoritative interlocutor may have been able to steer the discourse toward greater sophistication. Discursive entry points were identified in Excerpt B as well. But in the case of Sara and Lisa, the discussion orbited in a cyclical rut for a while before Lisa suggested they try a different approach. Unfortunately, the seminar drew to a close, and we have no way of ever knowing what the outcome of their discussion would have been if Sara and Lisa continued on their own. But in all actuality, the conditions of this scenario bear resemblance to those of proof construction by mathematicians—where circularity and open-ended uncertainty are the norm. This brings us to the crux of the issue.

Small group discourse for learning mathematical proof may be one option for teachers who wish to create a classroom environment modeled on characteristics of authentic mathematics. However, teachers operate under inescapable temporal conditions. How does a teacher balance the ideal with the confines of a *real* classroom? Moreover, there is the issue of the learning paradox. How do learners know (communicate about) what they do not know (communicate about), until an expert intervenes? Future discourse analysis, I believe, has the potential to address these issues. As dual teacher-researcher, I find the following particular questions intriguing.

- Are interlocutors more "receptive/primed" for learning/expert intervention *after* they have "struggled on their own" for a while?
- How long does an instructor let interlocutors struggle in their discourse before intervening?
- What types of intervention should the teacher provide in small-group discourse on mathematical proof?

Jacob's answer in the interview to what he didn't like about the seminar allowed me to probe him a bit further about *his* perspective as a student on the questions posed above. Here is the transcript from that portion of the interview.

Me:	What if anything didn't you like about this format? Or, if you have suggestions for improving it in the future.
Jacob:	It would have to be sometimes it was too up in the air, about where we had to go and hard to get the right track. Although once we got the right track.
Me:	When you were working in the small-groups?
Jacob:	Yea, when we were working in small-groups.
Me:	So can you think specifically how we would improve that?
Jacob:	Mmm
Me:	So not work in small-groups, or still work in small-groups but
Jacob:	Still work in small-groups because everybody gets engaged then and everybody has to not just rely on a couple people from the class for input.
Me:	Okay
Jacob:	I would have to say, it's a toughie, cause if you give the first couple of steps it's kind of obvious of where to go from there. A relatable example. One that's already solved but that's structured in kind of the same way, so you could kind of look if you need help to base it off of.
Me:	And should that one be done by the teacher in advance, or just one to look at?
Jacob:	One to look at, yeah, and maybe walk through before, just real quick though.
Me:	Is what I hear you saying, is that sometimes you were lost and lost for too long that it was unproductive?
Jacob:	Yea.
Me:	If . . . I could be in more places at one time, once I got there, did that, was that?
Jacob:	Oh yea, once you came and like helped us personally that did help us
Me:	Got you back on track?
Jacob:	Yea.
Me :	Okay. This is interesting to me in terms of learning. . .there's a balance here that I'm interested in. Do you think that it's important for students to kind of struggle for a little bit before they get that exact answer?
Jacob:	Oh yea.
Me:	Do you think you would have had as much understanding, like you said, if I just told you immediately what the steps were?
Jacob:	No way.
Me:	What do you think the balance should be?
Jacob:	Throw us to the wolves. And let us solve it for a little while. And then if you see puzzled faces then go over and ask, "Hey, how's it going?" And if it's not going so good then push them in the right direction.

Jacob's comments indicate that as a learner he values the struggle, but desires the support of an expert. The challenge for teachers (and future discursive researchers) is one of balance. It is a question of when to refrain from intervention, allowing learners to experience the authentic struggle of mathematics, and when intervention is most optimal for student learning.

The synthesis of the findings related to Research Question 1 illustrates the complexity of learning mathematical proof; that is, of becoming a more expert participant in the discourse of mathematical proof. Small-group discourse appears to be a comfortable way for novice interlocutors to *approach* a more expert discourse on proof. Moreover, there may exist in discourse between novice interlocutors natural and especially ripe opportunities, called *discursive entry points*, in which experts could intervene to steer the discourse towards increasing sophistication. Additionally, the study revealed several complex and interrelated factors related to novice interlocutors' communication (thinking) of mathematical proof. The factors include: discursive contributions/role of interlocutors, discursive foci of interlocutors, difficulty/familiarity of mathematical content, negotiating effective communication, commognitive conflict, and power. Let us now turn our focus to the question of *community*.

Research Question 2

The research design of this study allows for comment on the nature of a community built around a common academic purpose—the learning of mathematical proof. The zero-credit seminar utilizing mainly small-group discourse represented an attempt to grow a "creative intellectual social unit" within a mathematics department at a small Western Pennsylvania university (Carnegie Foundation, 1990, p. 13). Moreover, mathematics is defined as a specialized discourse. Thus, to learn mathematics (mathematical proof) is to become a more capable participant in its discourse community. What is the nature of undergraduate mathematics majors' sense of community as related to a seminar on mathematical proof utilizing a small-group discussion format? A seminar utilizing small-group discourse, it seems, plays an important social function. It serves to connect interlocutors first to one another within the seminar, on campus, and possibly even, as Jacob called it, to the "math world." The findings related to the second research question are now synthesized.

Community within Seminar

Off-task social interactions were an innate component of the small-group discourse on mathematical proof in the seminar. While frequent wanderings from the proof-tasks made for messy analysis of the transcripts, they represent important findings. These off-task interactions ranged from lighthearted moments about being mathematical marvels (Excerpt B) to frequent and sometimes-lengthy discussions about food, weather, other classes, and more. They were, in essence, opportunities for interlocutors to get to know each other. It seems that these social interactions may have been one element contributing to the reasonable sense of connectedness that participants developed in the seminar, as indicated by the *Classroom Community Scale*. Nine of ten items on the *Connectedness Subscale* received a mean rating of three or higher, with 4 the highest overall rating. The overall subscale mean rating was 31.375 (with 40 the highest possible rating). Participants especially felt that students in the seminar cared about each other (Item 1, M=3.5) and that they would be supported by their fellow classmates (Item 19, M =3.5).

The survey results corroborate with findings from the interview portion of the study. During the interviews, participants were asked to provide their own characterization of an undergraduate mathematical learning community. As was previously reported, the element of support largely pervaded the characterizations. When presented with the follow-up question of how closely the math seminar matched their own idea of a mathematics learning community, responses included: "really close," "pretty close," "probably well," "pretty close actually," "a little bubble of it," and "the basis for that interpretation." The overall *Classroom Community Scale* mean score of 61.750 (out of 80) indicated that the participants felt a reasonable sense of community *within* the seminar learning environment and that this sense was fairly balanced between *learning* (M=30.375) and *connectedness* (M=31.375).

Community on Campus

Sara described the seminar as a "little bubble" of an undergraduate mathematics community. "If you were to expand it [the seminar] that is what I would want out of an undergraduate math community." While they were perhaps immediately unproductive in terms of learning mathematical proof, the off-task interactions may have additionally contributed, in part, to a less overt and ongoing (unfinished) outcome of the seminar—the formation of a mathematical community that extended beyond the seminar walls. The interview portion of the study revealed that participants valued the opportunity to become better acquainted with their fellow math majors. The following four interview excerpts illustrate this.

Tracy

Tracy:	And being in that group, it helped talking with each other outside of the classroom. In our math classes we talked a lot more after the seminar.
Me:	So after the seminar you felt like you knew people better in your classes?
Tracy:	Exactly, yes, talked with them a lot better.

Me:	What do you think the impact of the seminar was on the overall math community here at our university?
Tracy:	Well yea, like I said, it helped talking to people in other math classes and just on campus overall. Like when you're walking across seeing each other on campus you say hi and stuff like that.

Nicole

Me:	What do you think the impact of the seminar was on any overall university math community, if any?
Nicole:	Well, you get to know other people that you didn't know before. Like the freshman. Like being in there, we got to meet the freshman. There's more freshman math majors than sophomores. So you get to meet more people definitely.

Lisa

Lisa:	I think it's important because I know there's not that many math majors up here. And so like knowing who the other math majors are helps. If I need help with something I can always go and ask them. I'll know who to ask, because I didn't know some of the people were math majors in there until I went into the class (seminar) So it definitely helped me.

Lisa:	I know I met with a few of the people in my class. A lot of the freshman I met outside of class. So I would say yea.
Me:	You might not have known them if you hadn't worked with them in seminar?
Lisa:	Right, some of them were in my class but I didn't really talk to them that much. But seeing them there, I had more of an opportunity to talk to them.
Me:	So you started to make more connections?
Lisa:	Right.

Me: With them because you saw them in seminar and then you were able to connect with them outside of seminar?
Lisa: Right.

Karen

Me: How closely did the seminar experience resemble your idea of what a math community should be? Is there a connection there?
Karen: When I got here, I didn't think it was going to be like that. But a math seminar makes you closer. And I think it helps because if I didn't have it I wouldn't know them as much as I do. And so I think it helps getting to know people in your major specifically. I think it's a good factor.

Springer, Stanne, and Donovan's (1999) meta-analysis demonstrated that small-group learning contributes to increased persistence through STEM (Science, Technology, Engineering and Mathematics). We can perhaps conjecture that the feelings of connectedness that grew out of and extended beyond the seminar may specifically factor into study participants' persistence in their future major coursework.

The Wider Mathematics Community

Finally, there was some indication that participants appreciated the seminar on mathematical proof utilizing small-group discourse as preparation for *communicating* in a more omnipresent "math world." Specifically, they viewed mathematical proof as fundamental to all of mathematics. Moreover, they recognized mathematical communication as both specialized and privileged. A few students referenced communication in a mathematics community that extended beyond their undergraduate education. Karen, for example, recognized proof almost as the hidden code to all of mathematics—a code that she felt was necessary for her to understand if she was to teach mathematics someday. Finally, Jacob felt that proof was a means of universal communication within a mathematics community.

Future Research

A main contribution of this study has been to uncover factors of, and issues related to, small-group discourse as it relates to the learning of mathematical proof. Each of the factors identified—discursive role/contributions of interlocutors, discursive foci of interlocutors, difficulty/familiarity of mathematical comment, navigating effective communication, commognitive conflict, and power—are in and of themselves research domains in need of further probing. Suggestions for the potential direction of research related to each of the factors were given in their respective discussions. Here, recommendations related to discursive entry points, research design, and research tools are discussed.

Discursive Entry Points: A Research Area with Promise

The findings of this study revealed the potential for expert intervention in naturally occurring openings in beginner discourse, called *discursive entry points.* The conception of discursive entry points seems especially promising in the current era of mathematics education reform, which calls for greater student engagement/conversation in the classroom. Greater student engagement/conversation in the classroom, however, requires that the educator strike a very fine balance. The educator must work to foster a learning environment that allows for the entire range of mathematical discovery (from the trials and tribulations of working on a proof to the elation of seeing it to completion). He or she must also adeptly manage the problem of the learning paradox. Thus, the task of the future discursive researcher will be two-fold. First, researchers will need to identify and classify points in student discourse that are pregnant with potential. Where in students' small-group discourse will expert intervention bear the greatest fruit? What opportunities present experts the greatest chance of guiding learners to increased proficiency in the discourse of mathematical proof? Second, discursive researchers will no doubt be interested in the question of *how*. How do experts intervene when presented with discursive entry points in small-group discourse and how do learners respond? The findings of future exploration into naturally occurring discursive entry points in student discourse could potentially transform the teaching and learning of mathematical proof.

Research Design

In this study, data were collected strictly from the natural setting. Other than the deliberate pairing of study participants (and non-participants), no attempt was made to manipulate the naturally-occurring conditions of the seminar. Because one of the goals of the seminar was to get fellow mathematics majors to know one another, pairings of students regularly changed. Moreover, given the informal nature of the zero-credit seminar, progress on the material was determined by the students' pace, not a pre-defined syllabus. Lastly, to provide students with support, I was a somewhat frequent participant in their small-group discourse. Investigation of discourse in the natural teaching and learning setting is invaluable as it reflects the true conditions of teaching and learning. But certainly, a more controlled non-classroom environment would also provide a researcher with affordances. For example, a researcher who observes and collects discourse of a single pair of students discussing proof with no expert intervention will no doubt be able to offer different findings on student learning. Moreover, a more controlled environment in which the researcher was observing participants at all times would allow for an in depth examination of mediation of discourse that extends beyond their verbal utterances. Of particular interest would be the body language of interlocutors—their physical gestures, facial expressions, posture, and pointing.

Future discursive research on the learning of mathematical proof by undergraduate math majors should encompass a variety of research designs. Sfard (2008) states:

> Discourses may be analyzed with respect to their inner dynamics, to the factors that make them change, to the roles of interlocutors, and so forth The phenomena under study may differ in their time scales, in their participants, in the context of their occurrence—and the list is long. (p. 276)

A wide range of discursive approaches to studying the learning of mathematical proof could include:

- *Longitudinal study of single pair of students learning mathematical proof.* In this study, we have defined mathematical learning as becoming fluent in discourse that is recognized as mathematical by experts. Researchers especially interested in how that fluency develops in students could undertake a longitudinal study of the discourse of a single pair of undergraduate students. In such a study, both how individual interlocutor's utterances change over time and how the pair's combined utterances change over time would be of utmost interest. Sfard and Kieran (2001), for example, analyzed data from a two-month-long series of interactions between two 13-year old boys learning algebra and found their communication to be ineffective throughout.

- *Compare/contrast multiple paired-discourses on a singular proof task.* Researchers interested in how students learn a particular type of proof technique or proof in a certain content area could consider comparing/contrasting multiple paired-discourses on a singular proof task. This might prove especially helpful in identifying patterns related to: 1) discursive entry points; 2) ad hoc routine courses of action; and 3) transitions by interlocutors in discursive foci. Combined, this information would be invaluable for instructors endeavoring to use small-group discourse in the teaching and learning of mathematical proof.
- *Effects of paired discourse on thinking about mathematical proof.* A legitimate pedagogical question concerns the impact of small-group discourse on individual thinking. Kieran (2002) studied this "co-shaping of public and private discourse, and some of the circumstances under which one occasions the other in the evolution of mathematical thinking" for pairs of 13-year olds who were graphing problem situations involving rational functions. The study's participants were asked to: 1) to collaborate, express ideas aloud, and assist one another in solving a fairly difficult multi-level problem; 2) write individual reports on what they had done with their partner; and 3) work individually on a problem comparable to the problem from step 1. Kieran compared ratings of individual data with ratings of the paired-discourse data to characterize how mutually productive the paired work was. A similar study design could be used with undergraduates and proof tasks.

The setting for the current study was an informal seminar in which students worked on introductory proof techniques. The proofs required a college preparatory math background (algebra, geometry, and trigonometry). Study participants consisted of freshman and sophomore mathematics majors, none of whom had yet to complete the calculus sequence. Certainly, this study only scratched the surface of a subject with infinite potential. Thus, future discursive research on undergraduate learning of mathematical proof will benefit from studies performed in other settings. These should include:

- *For-credit proof classes.* It seems reasonable to assume that the nature of paired-discourse as well as students' motivations and attitudes toward learning proof in small-group settings, and consequently their discourse, could differ in a for-credit class. For example, students might be less likely to engage in off-task discussion in a for-credit class, potentially resulting in more focused and perhaps more mathematically productive discourse. On the other hand, there may exist in for-credit classes a competitive atmosphere that did not seem to exist in the seminar in this study. Such an atmosphere may encourage some students to dominate discourse and others to shy away from it.
- *Classrooms utilizing whole-group discourse.* The focus of this study was on small-group discourse. However, as mathematics instructors at the university level begin to experiment with non-lecture formats (e.g., King, 2001), there will be much to gain from investigations into communication (thinking) on mathematical proof that occurs in whole-group contexts. This will require the development of new types of discursive analysis tools or a re-conception of Sfard and Kieran's (2001) focal and preoccupational analysis tools, which are designed for analyzing paired discourse.

- *Range of mathematical content/levels.* Unquestionably, the current study does not provide the last word on mathematics majors' beginning level thinking on mathematical proof. Thus, future research should build on this study in introduction-to-proof or bridge-to-advanced-mathematics type settings. However, extensive discursive studies on mathematics majors' proof work in *specific* content areas promises to be beneficial as well. Discursive research in calculus, linear algebra, Euclidean and non-Euclidean geometries, abstract algebra, number theory, and real-analysis settings may reveal insights that could radically change the teaching and learning of proofs in these subjects in the future.

Development of Data Analysis Tools

Discursive research on mathematical thinking is still in its infancy. As such, researchers will find themselves developing and adapting tools of analysis to fit their research needs. In this study, for example, my specific research curiosities led me to refine the preoccupational analysis tool. The reader is reminded that, in preoccupational analysis, one of the classification schemes for an interlocutor's utterance depends on whether the utterance develops the mathematical content of the discourse. Object-level utterances are "considered to be integral to moving the mathematical dimension of the discourse forward," whereas non-object-level utterances are those that "simply keep the conversation going, as well as those that reflect the relationship between interlocutors" (Kieran, 2002, p. 194). Kieran suggests that object-level utterances include reading the problem/text, rebutting a suggestion, offering a new suggestion, providing mathematical content, and seeking information of a mathematical nature. I propose that a type of object-level utterance specific to discourse on mathematical proof relates to what I have called *managing the proof process.* These object-level utterances may pertain, for example, to identifying the key question and its abstract answer or to coordinating a forward approach from the hypothesis with a backward approach from the conclusion. In my analysis, I used a colored-dotted arrow to represent "proof process" object-level utterances. The colored arrows proved particularly helpful in drawing attention to the types of contributions that each interlocutor made to the overall discourse. Moreover, the colored arrows have the potential to demarcate the discourse into smaller phases. However, this is not always the case. As we have seen, partners do not always carefully analyze the suggestions in the proof process. Finally, like Excerpt C, not all paired-discourses on mathematical proof will contain "proof process" object-level utterances. A sample of the proccupational analysis from Excerpt B is shown in Figure 41 to illustrate the just-discussed adaption to the preoccupational analysis tool. The "proof process" object-level utterances have been shaded rather than colored here.

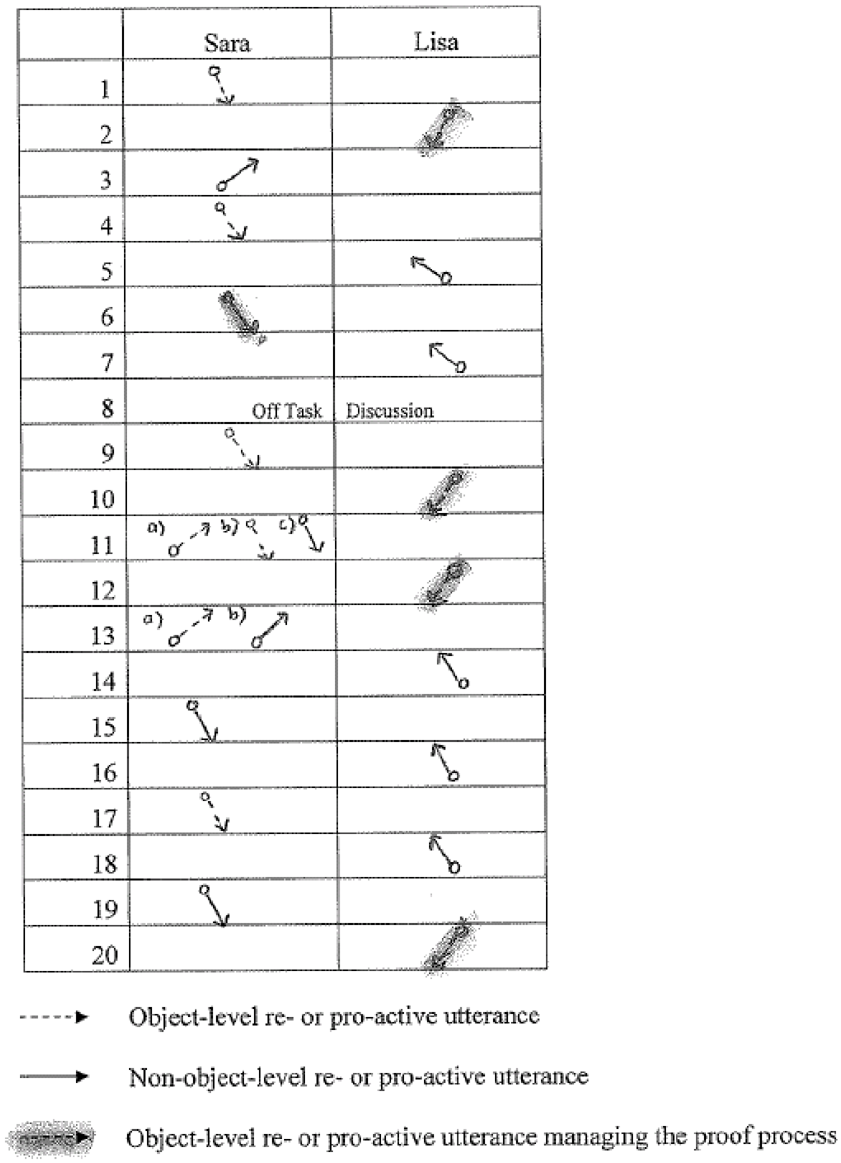


Figure 41. Example of modified preoccupational analysis.

Implications for Teaching

Teacher Understanding of Student Learning

Researchers of K-12 mathematics education are investigating more and more the link between teacher understanding of students' mathematical thinking and effective teaching practice. Steinberg, Empson, and Carpenter (2004), for example, report on the generative nature of one elementary school mathematics teacher's learning about children's thinking. As the teacher's instructional practices were increasingly guided by knowledge of children's thinking, her level of engagement with children's thinking increased (as measured by Franke, Carpenter, Levi, and Fennema's, 2001 scale), and vice versa. Doerr (2006) documents how one secondary teacher listened to students' alternative solution strategies on a well-known multiplicative growth problem and how she responded to those strategies in practice. Specifically, when the teacher listened to the students' strategies for the explicit purpose of understanding their thinking, rather than for evaluating their ideas, it led the students to greater engagement with the task and refinement of their own thinking.

A focused practice of teacher examination of students' learning of mathematical proof at the collegiate level may also have generative power in the classroom. While the theoretical framework underlying the focal and preoccupational analysis tools is highly complex, the tools themselves are unsophisticated. They provide a simple structure to support potentially powerful examination of student thinking (communication). Armed with the rudiments of their use, college instructors could utilize the focal and preoccupational analysis tools on samples of their students' discourse to improve their understanding of students' thinking on mathematical proof. Implementation of this professional development type activity could draw on related research, such as the study by Heid, Blume, Zbiek and Edwards (1999) on the practice of teaching teachers how to do interviews to understand their students' mathematical understandings. The current study points to the benefit that college instructors might derive from engaging in such an activity, as we repeatedly saw that a close examination of what students said (transcript) and did (work) was far more revealing than their work alone could ever be.

The Complexities of Teacher Practice

Fostering classroom environments in which students actively talk about mathematics is at the core of K-12 mathematics education reform (National Council of Teachers of Mathematics, 2000). In higher education, there exists similar interest (Barker et al., 2004; Berry & Sharp, 1999; Dannels, 2000; King, 2001; Neal, 2008; Northedge, 2003; Nunn, 1996). Researchers and authors focusing on K-12 mathematics education have offered descriptions of mathematics discourse communities (Hufferd-Ackles et al., 2004), models for fostering classroom discourse (Manouchehri & Enderson, 1999; Truxaw & De Franco, 2007; Van Zoest & Enyart, 1998), and discussions related to discourse and instruction (Manouchehri, 2007). Yet the actual implementation (teaching) and outcomes (learning) of increased student discourse in the mathematics classroom are exceptionally complex.

From a two-month-long study of the discursive interactions of two 13-year old boys who were learning algebra, Sfard and Kieran (2001), for example, found that the “merits of learning-by-talking cannot be taken for granted” (p. 42). The detailed commognitive analysis revealed that any algebraic growth made over the two months by the partners could not be directly attributed to their collaboration. It is likely that the progress made by the more knowledgeable of the two partners would have occurred without the partner. In fact, Sfard and Kieran hypothesized that the partnership may have even hindered his progress. Overall, the combined focal and preoccupational analyses by the researchers pointed to ineffective communication as the reason for the generally unprofitable collaboration. In particular, the more knowledgeable partner was either unable or unmotivated to communicate his knowledge in a way that benefited the other.

Sherin (2002) presents an in-depth look at the “pedagogical tensions” that occurred in one middle-school mathematics teacher’s yearlong deliberate attempt to use student-centered discourse in a whole-class setting. In particular, the teacher found himself trying to balance the student-centered *process* of mathematical discourse with substantial mathematical *content.* He struggled with “trying to use students’ ideas as the basis for class discussion while also ensuring that the discussion is productive mathematically” (p. 205). Overall, the struggle resulted in a fluctuation of emphasis on process or content throughout the school year. For example, the researchers classified classroom discourse as high process/low content during the beginning of the school year, but low process/ high content at the end of the school year. Nathan and Knuth (2003) documented similar findings in a study of one middle-school teacher who, over a two-year period, worked to change her practices, to better reflect reform-based mathematics instruction. Specifically, as the teacher “removed herself as the analytic center [of the classroom] to invite greater student participation. . . student-led discussion increased manifold, but lacked the mathematical precision offered previously by the teacher” (p. 175). One method, however, used by the teacher in Sherin’s study to manage these contending elements of classroom discourse was a “filtering approach.” This approach included solicitation of multiple ideas from students to facilitate the process of student-centered mathematical discourse; encouragement of students to elaborate on their thinking and to compare and contrast their ideas with others suggested; focusing of students’ attention on a subset of mathematical ideas raised to highlight important content; and finally encouragement of student discourse on the significant mathematical ideas.

The studies of Sfard and Kiearn (2001), Nathan and Knuth (2003), and Sherin (2002) provide a backdrop for the implications for teaching and learning of the current study. On the one hand, small-group discourse appears to be a viable pedagogical choice (teacher discourse move) for college mathematics instructors looking to increase student participation in academic discourse. Moreover, unlike a traditional lecture, such a format may more closely approximate the conditions of genuine mathematical activity. Instructors who choose to use small-group discourse to teach mathematical proof then, are emphasizing the student-centered *process* of discourse. However, similar to the results of the study of the paired discourse of 13-year old boys learning algebra (Sfard and Kieran, 2001), the findings of this study suggest that "the merits of learning-mathematical-proof-by-talking cannot be taken for granted." The study revealed a number of complex factors that learners need to negotiate to become increasingly proficient in discourse recognized as mathematical. Moreover, it seems highly improbable that novice interlocutors will successfully negotiate the sum of these factors together without some intervention by an expert in the discourse.

Sfard (2008) argues: "scaffolded individualization is the only way for a 'newcomer' to enter a discourse governed by rules different from those that regulated her communicational activity so far. Individualization, by definition, requires proactive participation—and help—of this discourse's 'oldtimers'" (p. 282). The filtering approach is one model that teachers (oldtimers) can utilize during whole-group discourse to help learners (newtimers) individualize mathematical discourse. College teachers of mathematics will need similar models if small-group discourse is to be used effectively in introducing students to mathematical proof. In naming discursive entry points and identifying factors that affect small-group discourse on mathematical proof, this study has laid the groundwork for further research and development of such models. Although much work remains, teachers might use the findings from the study to test and revise their own assumptions, beliefs, and informal models about using small-group discourse to teach mathematical proof.

Conclusion

Solow (2005) opens his text with a "preface to the instructor." Here, he captures the all too frequent general milieu of the teaching and learning of mathematical proof.

> The inability to communicate proofs in an understandable manner has plagued students and teachers in all branches of mathematics. The result has been frustrated students, frustrated teachers, and, oftentimes, a watered-down course to enable the students to follow at least some of the material, or a test that protects students from the consequences of this deficiency in their mathematical understanding. (p. xiii)

This state of affairs is at best unfortunate and at worst tragic. Mathematical proof, after all, is the backbone of all of mathematics. Moreover, what the above-described scenario does not portray is a learning situation in which students taste the "experience and joy of mathematical discovery" (Benson, 1999). Benson argues that the joy of discovery should be just as intense for the learner of proof as it was for the first mathematician who discovered it.

The present study, I believe, presents a complex, but nonetheless hopeful, outlook on the teaching and learning of mathematical proof. Classroom methods that are dialogic in nature, such as small-group discourse, may hold the promise of promoting mathematical experiences for learners that are more authentic in nature—where the joy of mathematical discovery, if even on a small scale, is not uncommon. We are reminded here of Lisa's elation in Excerpt D when she sees the connection between the forward and backward steps of her proof. "I did it!" [256]. At the same time that dialogic methods of teaching mathematics gain increasing traction, we have in Sfard's (2008) *commognition* a complementary, not to mention revolutionary, foundation for researching mathematical thinking. This study has shown that the high-resolution analysis of student discourse reveals far more about student thinking on mathematical proof than could ever be found by looking, for example, at student work alone. As a first of its kind, this study represents the tip of the iceberg when it comes to using discourse analysis to better our understanding of how college level students learn mathematical proof and consequently how to teach mathematical proof.

References

Baker, D. & Campbell, C. (2004). Fostering the development of mathematical thinking: Observations from a proofs course. *PRIMUS: Problems, Resources, and Issues in Mathematics Undergraduate Studies, 14*(4), 345-353.

Barker, W., Bressoud, D., Epp, S., Ganter, S., Haver, B. & Pollatsek, H. (2004). *Undergraduate Programs and Courses in the Mathematical Sciences: CUPM Curriculum Guide 2004.* Washington, DC: The Mathematical Association of America.

Bauersfeld, H. (1988). Interaction, construction, and knowledge: Alternative perspectives for mathematics education. In D.A. Grouws, T.J. Cooney & D. Jones (Eds.), *Perspectives on research on effective mathematics teaching* (pp. 27-46).

Benson, D.C. (1999). *The moment of proof: Mathematical epiphanies.* New York: Oxford University Press.

Berger, P. (1996). Mathematics versus computer science: teachers' views on teacher roles and the relation of both subjects. In E. Pehkonen (Ed.), *Proceedings of the MAVI-3 Workshop,* Research Report 170, University Helsinki.

Berry, J. & Sharp, J. (1999). Developing student-centered learning in mathematics through co-operation, reflection and discussion. *Teaching in Higher Education, 4*(1), 27-41.

Bogdan, R.C. & Biklen, S.K (1992). *Qualitative research for education* (2^{nd} ed.). Boston: Allyn and Bacon.

Brabenec, R.L. (2001). A mathematics colloquium for sophomores. *PRIMUS: Problems, Resources, and Issues in Mathematics Undergraduate Studies, 11*(3), 197-208.

Carnegie Foundation for the Advancement of Teaching. (1990). *Campus life: In search of community.* Lawrenceville, NJ: Princeton University Press.

Cobb, P. (1994). Where is the mind? Constructivist and sociocultural perspectives on mathematical development. *Educational Researcher, 23*(7), 13-20.

Cobb, P. & Bauersfeld, H. (1995). Introduction: The coordination of psychological and sociological perspectives in mathematics education. In P. Cobb & H. Bauersfeld (Eds.), *The emergence of mathematical meaning* (pp. 1-16). Hillsdale, N.J.: Lawrence Erlbaum Associates.

Corbin, J. & Strauss, A. (2008). *Basics of qualitative research* (3rd ed.). Los Angeles: Sage Publications.

Cotton, K. (1989). *Classroom questioning.* Retrieved July 6, 2008, from http://www.nwrel.org/scpd/sirs/3/cu5.html

Crain, W. (2005). *Theories of development: Concepts and applications* (5th ed.). Upper Saddle River, N.J.: Prentice Hall.

Dannels, D.P. (2000). Learning to be professional: Technical classroom discourse, practice and professional identity construction. *Journal of Business and Technical Communication, 14*(1), 5-37.

Dean, E.E. (1996). Teaching the proof process. *College Teaching, 44*(2), 52-55.

Dillon, J.T. (1990). *The practice of questioning.* New York: Routledge.

Doerr, H. M. (2006). Examining the tasks of teaching when using students' mathematical thinking. *Educational Studies in Mathematics, 62*, 3-24.

Doise, W & Mugny, G. (1979). Individual and collective conflicts of centrations in cognitive development. *European Journal of Psychology, 9,* 105-108.

Doise, W., Mugny, G., & Perret-Clermont, A..N. (1975). Social interaction and the development of cognitive operations. *European Journal of Psychology, 5*(3), 367-383.

Ernest, P. (1994). Conversation as a metaphor for mathematics learning. *Proceedings of the British Society for Research into Learning Mathematics Day Conference, Manchester Metropolitan University. Nottingham: BSRLN,* 58-63.

Ernest, P. (1998). *Social constructivism as a philosophy of mathematics.* Albany: State University of New York Press.

Felix, U. (2005). E-learning pedagogy in the third millennium: the need for combining social and cognitive constructivist approaches. *ReCALL, 17*(1), 85-100.

Fleron, J.F. & Hotchkiss, P.K. (2001). First-year and senior seminars: Dual seminars= stronger mathematics majors. *PRIMUS: Problems, Resources, and Issues in mathematics Undergraduate Studies, 11*(4), 289-325.

Franke, M.L., Carpenter, T.P., Levi, L. & Fennema, E. (2001). Capturing teachers' generative change: A follow-up study of professional development in mathematics. *American Educational Research Journal, 38*, 653-689.

Gold, B. (1999). *Social constructivism as a philosophy of mathematics.* [Review of the books *Social constructivism as a philosophy of mathematics* and *What is mathematics really?*]. *American Mathematical Monthly, 106* (4), 373-380.

Grady, J. (1998). Persistence in science of high-ability minority students: Results of a longitudinal study. *The Journal of Higher Education, 69*(6), 589-620.

Greenberg, M.J. (1993). *Euclidean and non-Euclidean geometries: Development and history.* New York: W.H. Freeman.

Hale, M. (2003). *Essentials of mathematics: Introduction to theory, proof, and the professional culture.* Washington, DC: Mathematical Association of America.

Hatch, A. J. (2002). *Doing qualitative research in education settings.* Albany, NY: SUNY Press.

Heid, M.K., Blume, G.W., Zbiek, R.M., & Edwards, B.S. (1999). Factors that influence teachers learning to do interviews to understand students' mathematical understandings. *Educational Studies in Mathematics, 37,* 223-249.

Hersch, R. (1997). *What is mathematics really?* New York: Oxford University Press.

Honderich, T. (Ed.). (1995). *The Oxford companion to philosophy.* New York: Oxford University Press.

Hufferd-Ackles, K., Fuson, K.C. & Sherin, M.G. (2004). Describing levels and components of a math-talk learning community. *Journal for Research in Mathematics Education, 35*(2), 81-116.

Johnson, D. W., Johnson, R.T. , & Smith, K.A. (1991). *Active learning: Cooperation in the college classroom.* Edina, MN.: Interaction Book Company.

Kieran, C. (2002). The mathematical discourse of 13-year-old partnered problem solving and its relation to the mathematics that emerges. In C. Kieran, E. Forman, & A. Sfard (Eds.), *Learning discourse: Discursive approaches to research in mathematics education* (p. 187-228). Boston: Kluwer.

King, D.L. (2001). From calculus to topology: Teaching lecture-free seminar courses at all levels of the undergraduate mathematics curriculum. *PRIMUS: Problems, Resources and Issues in Mathematics Undergraduate Studies, 11*(3), 209-227.

Knuth, E. & Peressini, D. (2001). A theoretical framework for examining discourse in mathematics classrooms. *Focus on Learning Problems in Mathematics, 23*(2 & 3), 5-22.

Krussel, L., Edwards, B. & Springer, G.T. (2004). The teacher's discourse moves: A framework for analyzing discourse in mathematics classrooms. *School Science and Mathematics, 104*(7), 307-312.

Lakatos, I. (1976). *Proofs and refutations: The logic of mathematical discovery.* Cambridge: Cambridge University Press.

Lave, J. & Wenger, E. (1991). *Situated learning: Legitimate peripheral participation.* Cambridge: Cambridge University Press.

Lenning, O.T. & Ebbers, L.H. (1999). *The powerful potential of learning communities: Improving education for the future.* ASHE-ERIC Higher Education Report Volume 26, No. 6. Washington, D.C.: The George Washington University Graduate School of education and Human Development.

Lincoln, Y.S. & Guba, E.G. (1985). *Naturalistic inquiry.* Newbury Park: SAGE Publications.

Lutzer, C.V. (2005). The role of proof in mathematics and mathematics education. *PRIMUS: Problems, Resources, and Issues in mathematics Undergraduate Studies, 15*(4), 374-384.

Manouchehri, A. (2007). Inquiry-discourse: Mathematics instruction. *Mathematics Teacher, 101*(4), 290-300.

Manouchehri, A. & Enderson, M.C. (1999). Promoting mathematical discourse: Learning form classroom examples. *Mathematics Teaching in the Middle School, 4*(4), 216-222.

Martin, J. & Sugarman, J. (1997). The social-cognitive construction of psychotherapeutic change: Bridging social constructionism and cognitive constructivism. *Review of General Psychology, 1*(4), 375-388.

May, G.S. & Chubin, D.E. (2003). A retrospective on undergraduate engineering success of underrepresented minority students. *Journal of Engineering Education, 92*(1), 27-39.

Mehan, H. (1979). *Learning lessons: Social organization in the classroom.* Cambridge: Harvard University Press.

Merriam, S. B. (1988). *Case study research in education: A Qualitative Approach.* San Francisco: Jossey-Bass.

Merriam, S. B. & Associates. (2002). *Qualitative research in practice: Examples for discussion and analysis.* San Francisco: Jossey-Bass.

Moore, R.C. (1994). Making the transition to formal proof. *Educational Studies in Mathematics, 27*(3), 249-266.

Nathan, M. J.& Knuth, E.J. (2003). A study of whole classroom mathematical discourse and teacher change. *Cognition and Instruction, 21*(2), 175-207.

National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics.* Reston, VA: National Council of Teachers of Mathematics.

Neal, M. (2008). Look who's talking: Discourse analysis, discussion, and initiation-response-evaluation patterns in the college classroom. *Teaching English in the Two Year College, 35*(3), 272-281

Noddings, N. (1990). Constructivism in mathematics education. In R.B. Davis, C.A. Maher, & N. Noddings (Eds.), *Constructivist views on the teaching and learning of mathematics Monograph no. 4*(pp. 7-18). Reston, VA: national Council of Teachers of Mathematics

Northedge, A. (2003). Enabling participation in academic discourse. *Teaching in Higher Education, 8*(2), 169-180.

Nunn, C.E. (1996). Discussion in the college classroom: Triangulating observational and survey results. *Journal of Higher Education, 67*(3), 243-266.

Ornstein, A. C. & Hunkins, F.P. (2004). *Curriculum: Foundations, principles and issues* (4th ed.). New York: Pearson.

Palincsar.A.S. (1998). Social constructivist perspectives on teaching and learning. *Annual Review of Psychology, 49*, 345-375.

Palmer, P.J. (1998). *The courage to teach: Exploring the inner landscape of a teacher's life.* San Francisco: Jossey-Bass.

Patton, M.Q. (1990). *Qualitative evaluation and research methods* (2nd ed.). Newbury Park: Sage Publications.

Patton, M.Q. (2002). *Qualitative research & evaluation methods* (3rd ed.). Thousand Oaks, CA: Sage Publications.

Phillips, D.C. (1995). The good, the bad, and the ugly: The many faces of constructivism. *Educational Researcher, 24*(7), 5-12.

Piaget, J. (1970). *Genetic epistemology.* New York: Columbia University Press.

Piaget, J. (1971). *Biology and knowledge.* Edinburgh, UK: Edinburgh Press.

Piaget , J. (1985). *The equilibration of cognitive structures: The central problem of intellectual development.* Chicago: University of Chicago Press.

Polya, G. (1954). *Mathematics and plausible reasoning (Vol. 1).* Princeton, N.J.: Princeton University Press.

Rosenberg, M. (1983). *An introduction to sociology.* Ontario: Routledge.

Rovai, A. P. (2002). Development of an instrument to measure classroom community. *Internet and Higher Education, 5*, 197-211.

Sfard, A. (2002). There is more to discourse than meets the ears: Looking at thinking as communicating to learn more about mathematical learning. In C. Kieran, E. Forman, & A. Sfard (Eds.), *Learning discourse: Discursive approaches to research in mathematics education* (p. 13-57). Boston: Kluwer.

Sfard, A. (2008). *Thinking as communicating: Human development, the growth of discourses, and mathematizing.* New York: Cambridge University Press.

Sfard, A., Forman, E. & Kieran, C. (2002). Guest editorial. In C. Kieran, E. Forman, & A. Sfard (Eds.), *Learning discourse: Discursive approaches to research in mathematics education* (p. 1-11). Boston: Kluwer.

Sfard, A. & Kieran, C. (2001). Cognition as communication: Rethinking learning-by-talking through multi-faceted analysis of students' mathematical interactions. *Mind, Culture, and Activity, 8*(1), 42-76.

Sherin, M.G. (2002). A balancing act: Developing a discourse community in a mathematics classroom. *Journal of Mathematics Teacher Education, 5*, 205-233.

Smith, B.L., MacGregor, J., Mathhews, R.S., & Gabelnick, F. (2004). *Learning communities: Reforming undergraduate education.* San Francisco: Jossey-Bass.

Smith, D.G. (1977). College classroom interactions and critical thinking. *Journal of Educational Psychology, 69*(2), 180-190.

Solomon, Y. (1989). *The practice of mathematics.* London: Routledge.

Solow, D. (2005). *How to read and do proofs* (4th ed.). New York: John Wiley & Sons.

Springer, L., Stanne, M.E., & Donovan, S.S. (1999). Effects of small-group learning on undergraduates in science, mathematics, engineering, and technology: A meta-analysis. *Review of Educational Research, 69*(1), 21-51.

Sriraman, B. (2004). The characteristics of mathematical creativity. *The Mathematics Educator, 14*(1), 19-34.

Steen, L.A. (1990). Pattern. In L.A. Steen (Ed.), *On the shoulders of giants: New approaches to numeracy.* Washington, D.C.: National Academy Press.

Steinberg, R.M., Empson, S.B., & Carpenter, T.P. (2004). Inquiry into children's mathematical thinking as a means to teacher change. *Journal of Mathematics Teacher Education, 7,* 237-267.

Teppo, A. R. (1998). Diverse ways of knowing. In A.R. Teppo (Ed.), *Qualitative research methods in mathematics education* (pp. 1-16). Reston, VA: National Council of Teachers of Mathematics.

Terenzini, P.T., Theophilides, C., & Lorang, W.G. (1984). Influences on students' perceptions of their academic skill development during college. *The Journal of Higher Education, 55*(5), 621-636.

Truxaw, M.P. & DeFranco, T.C. (2007). Lessons from Mr. Larson: An inductive model of teaching for orchestrating discourse. *Mathematics Teacher, 101*(4), 268-272.

Tsui, L, (2002). Fostering critical thinking through effective pedagogy: Evidence from four institutional case studies. *The Journal of Higher Education, 73*(6), 740-763.

Van Zoest, L.R. & Enyart, A. (1998). Discourse of course: Encouraging genuine mathematical conversations. *Mathematics Teaching in the Middle School, 4*(3), 150-157.

von Glasersfeld, E. (1987). Learning as a constructive activity. In C. Janvier (Ed.), *Problems of representation in the teaching and learning of mathematics (*pp. 3-17). Hillsdale, NJ: Lawrence Erlbaum.

Vygotsky, L.S. (1978). *Mind in society: the development of higher psychological processes.* Cambridge, MA: Harvard University Press.

Walters, N.B. (1997). *Retaining aspiring scholars: Recruitment and retention of students of color in graduate and professional science degree programs.* Paper presented at the meeting of the Association for the Study of Higher Education, Albuquerque, NM. (ERIC Document Reproduction Service No. ED415816)

Weber, K. (2001). Student difficulty in constructing proofs: The need for strategic knowledge. *Educational Studies in Mathematics, 48,* 101-119.

Weber, K. (2008). How mathematicians determine if an argument is a valid proof. *Journal for Research in Mathematics Education, 39*(4), 431-459.

Weimer, M. (2002). *Learner-centered teaching: Five key changes to practice.* San Francisco: Jossey-Bass.

Wiersma, W. (2000). *Research methods in education: An introduction.* Boston: Allyn and Bacon.

Wilson, B.G., Ludwig-Hardman, S., Thornam, C.L., & Dunlan, J.C. (2004, November). Bounded community: Designing and facilitating learning communities in formal courses. *International Review of Research in Open and Distance Learning 5*(3). Retrieved July 20, 2008, from http://www.irrodl.org/index.php/irrodl/issue/view/19

Wittgenstein, L. (1997). *Tractatus logico-philosophicus.* (D.F. Pears & B.F. McGuinness, Trans.). New York: Routledge. (Original work published in 1921)

Yackel, E. & Cobb, P. (1996). Sociomathematical norms, argumentation, and autonomy in mathematics. *Journal for Research in Mathematics Education, 27*(4), 458-477.

Appendix A

Classroom Community Scale

CCS Test Booklet

Classroom Community Scale (CCS)

Developed by
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SURVEY

DIRECTIONS: Below you will see a series of statements concerning a specific course or program you are presently taking or recently completed. Read each statement carefully and place an X in the parentheses to the right of the statement that comes closest to indicate how you feel about the course or program. You may use a pencil or pen. There are no correct or incorrect responses. If you neither agree nor disagree with a statement or are uncertain, place an X in the neutral (N) area. Do not spend too much time on any one statement, but give the response that seems to describe how you feel. ***Please respond to all items***

Strongly Agree (SA) Agree (A) Neutral (N) Disagree (D) Strongly Disagree (SD)

1. I feel that students in this course care about each other............................(SA) (A) (N) (D) (SD)
2. I feel that I am encouraged to ask questions..(SA) (A) (N) (D) (SD)
3. I feel connected to others in this course...(SA) (A) (N) (D) (SD)
4. I feel that it is hard to get help when I have a question ....(SA) (A) (N) (D) (SD)
5. I do not feel a spirit of community..(SA) (A) (N) (D) (SD)
6. I feel that I receive timely feedback..(SA) (A) (N) (D) (SD)
7. I feel that this course is like a family ....(SA) (A) (N) (D) (SD)
8. I feel uneasy exposing gaps in my understanding ....(SA) (A) (N) (D) (SD)
9. I feel isolated in this course ....(SA) (A) (N) (D) (SD)
10. I feel reluctant to speak openly ....(SA) (A) (N) (D) (SD)
11. I trust others in this course..(SA) (A) (N) (D) (SD)
12. I feel that this course results in only modest learning.............................(SA) (A) (N) (D) (SD)
13. I feel that I can rely on others in this course ....(SA) (A) (N) (D) (SD)
14. I feel that other students do not help me learn ....(SA) (A) (N) (D) (SD)
15. I feel that members of this course depend on me....................................(SA) (A) (N) (D) (SD)
16. I feel that I am given ample opportunities to learn ....(SA) (A) (N) (D) (SD)
17. I feel uncertain about others in this course ....(SA) (A) (N) (D) (SD)
18. I feel that my educational needs are not being met.................................(SA) (A) (N) (D) (SD)
19. I feel confident that others will support me ....(SA) (A) (N) (D) (SD)
20. I feel that this course does not promote a desire to learn........................(SA) (A) (N) (D) (SD)

Scoring Key

Overall CCS Raw Score

CCS raw scores vary from a maximum of 80 to a minimum of zero. Interpret higher CCS scores as a stronger sense of classroom community.

Score the test instrument items as follows:

For items: 1, 2, 3, 6, 7, 11, 13, 15, 16, 19
Weights: Strongly Agree = 4, Agree = 3, Neutral = 2, Disagree = 1, Strongly Disagree = 0

For items: 4, 5, 8, 9, 10, 12, 14, 17, 18, 20
Weights: Strongly Agree = 0, Agree = 1, Neutral = 2, Disagree = 3, Strongly Disagree = 4

Add the weights of all 20 items to obtain the overall CCS score.

CCS Subscale Raw Scores

CCS subscale raw scores vary from a maximum of 40 to a minimum of zero. Calculate CCS subscale scores as follows:

Connectedness	Add the weights of odd items: 1, 3, 5, 7, 9, 11, 13, 15, 17, 19
Learning	Add the weights of even items: 2, 4, 6, 8, 10, 12, 14, 16, 18, 20

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Appendix B

Permission to use Classroom Community Scale

Mail Message

[Mail](#) [Properties](#)

From:	Alfred Rovai <alfrrov@regent.edu>	Sunday - July 20, 2008 7:38 PM
To:	Katherine Remillard <kremillard@francis.edu>	
Subject:	RE: Sense of Classroom Community Index--Permission Request	
Attachments:	I&HE4.PDF (111287 bytes)	[View] [Open] [Save As]
	CCS Booklet.pdf (122568 bytes)	[View] [Open] [Save As]
	Mime.822 (325051 bytes)	[View] [Save As]

Hi Kate,

Sure you may use the instrument, just cite the attached article. I've also attached a clean copy of the instrument.

Best wishes,
Fred

Alfred P. Rovai, PhD
Professor of Education
Regent University
1000 Regent University Drive
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http://www.regent.edu/acad/schedu/pdfs/vita_rovai.pdf

Appendix C

Guiding Questions for Interview

Thank you very much for your participation in the research study throughout the past semester and in this interview this evening/afternoon. I want to remind you that your participation is voluntary and all of the data collected last semester and in the interview this evening/afternoon is confidential. Pseudonyms will be used for any data that is reported on. I have the original consent form here that you signed last semester. Would like to review it? Are you still agreeable to me audio recording our interview?

Your answers to the questions will be most helpful if they are *honest* and *complete.* In no way will your answers affect your grades, status within the mathematics department/university, or services provided to you by the mathematics department/university.

1. Several features about the freshman/sophomore math seminar last semester were different from typical college math classes at this university. (No homework problems , no final grade , attended by only math majors, limited teacher talk, informal opportunity to work in small groups and with instructor)

    a. What, if anything, did you like about this format?
    b. What, if anything, didn't you like about this format?
    c. What suggestions would you make for improving the format in the future?
    d. How did this format, specifically the opportunity to work in small groups with other majors and receive my feedback, impact your *learning* of mathematical proof, both positively or negatively?
    e. The next question is specific to learning/working on mathematical *proof* with a partner. When working with a partner, in general, to what extent (if any) did you find hearing your partner's ideas/sharing and getting feedback on your ideas helpful? And to what extent (if any) did hearing your partner's ideas/having them give feedback on your ideas make thinking/working on the proof more difficult?
2. In the university, there are many communities to become a part of (athletics, fraternities/sororities, dorms, drama, pep band, student government, student activities, clubs etc.)
    a. In your opinion, how important, or is it important at all, for a math major to feel or become part of a mathematical community?
    b. Why do you feel this way?
    c. What do you think an undergraduate mathematical community should look or feel like?
    d. How closely did seminar resemble your idea of an undergraduate mathematical community?
    e. What do you think the impact of the seminar was on the overall university mathematical community?

3. Seminar could have been used to explore any number of mathematical topics.
    a. What, if any, was the value of working specifically on mathematical proof?
    b. What, if any, connection do you see to specifically learning about proof in a seminar format and being part of a mathematical learning community?

4. Would you like to share any other insights/comments/thoughts about mathematical proof, seminar, or mathematical communities?

Appendix D

Reference on Basic Proof Terminology and Strategies

I. Basics

Conditional statement (also know as an implication) A statement of the form "If *A*, then *B*" (or *A* implies *B*). The symbolic representation of the conditional is $A \rightarrow B$.

- Statement *A* is called the *hypothesis*. It is also referred to as the *antecedent* or the *given.*
- Statement *B* is called the *conclusion*. It is also referred to as the *consequent*.

The goal of mathematical proof is to develop a convincing argument that "If *A* is true, then *B* is true."

II. Strategies/Terminology (based on Daniel Solow's *How to Read and Do Proofs* 4th ed.)

Key question The specific question obtained by asking how you can show that a given statement *B* is true. A properly posed key question should contain no symbols or other notation (except for numbers) from the specific problem under consideration.

Abstract answer An answer to the key question that contains no symbols from the specific problem.

Applied answer The abstract answer applied to the specific problem using specific notation.

Forward-backward method The technique for proving that "*A* implies *B*" in which you assume that *A* is true and try to show that *B* is true. To do so, you apply the forward process to *A* and the backward process to *B*.

Forward process The process of deriving from a statement *A*, a new statement *A1*, with the property that *A1* is true because *A* is true.

Backward process The process of deriving from a statement *B*, a new statement, *B1*, with the property that if *B1* is true, then so is *B*. You do this by asking and answering a key question.

Reasons (justifications) Typically each statement arrived at in the forward and backward processes (*A, A1, A2. . . B2, B1, B*) is accompanied by a reason/justification statement. The reasons may include definitions, previously proven theorems, algebraic maneuvers etc. The statements (*A, A1, A2. . . B2, B1, B*) and their accompanying reasons make up the *analysis* of the proof.

Condensed proof (paragraph proof) The final version of a proof in paragraph form. It rarely contains the entire thought process that went into the proof. As a rule of thumb, the proof should contain sufficient detail to convince the person(s) to whom it is addressed (audience).

III. Example (Proposition 2 in Solow's text p. 25-26)

Conditional Statement:
If n is an even integer, then n^2 is an even integer.

Hypothesis (A):
n is an even integer

Conclusion (B):
n^2 is an even integer

Key Question:
How do I show a number is even?

Abstract Answer:
Show the number is equal to the product of two and an integer

Applied answer:
Show $n^2 = 2l$ where l is an integer

Proof Analysis:

Statement	**Reason**
A: n is an even integer	Given
A1: $n=2k$ where k is an integer	Definition of even
A2: $(n)(n) = (2k)(2k)$	Square both sides of equation in A1
A3: $n^2 = 4k^2$	Simplify A2
A4: $n^2 = 2(2k^2)$	Factor A3
B1: $n^2 = 2l$	Let $l = 2k^2$
B: n^2 is an even integer	From B1 and definition of even

Condensed Proof: Given n is an even integer, there exists an integer k such that $n=2k$. As a result, $n^2 = 4k^2 = 2(2k^2)$, thereby making n^2 even.

Appendix E

Transcript for Excerpt A

1	Sherri	They're like too simple
2	Jacob	How do we show a triangle is right? How do we show a triangle?
3	Sherri	Are you serious?
4	Jacob	a) That's really dumb and really obvious. No, I mean (pause) b) I like how do you show the area of a triangle.
5	Sherri	(to self as writing) a)How do you show area of a triangle? b) Abstract answer.
6	Jacob	We can't answer with a question right? I mean, we can't answer with an equation right?
7	Sherri	a) No (Jacob: No) b) Okay, we have
8	Jacob	Alright we're just going to (solve/simplify?) Jump right in?
9	Sherri	A. We know Pythagorean theorem which says (N: a squared) x squared plus your thingie equals z squared. Okay, so now those are equal
10	Jacob	How do you know z's the hypotenuse? Is that how it always is?
11	Sherri	Cause it tells you shows it shows you the figure (in an exasperated tone).
12	Jacob	Oh right, I'd forgotten about the figure good call
13	Sherri	dah
14	Sherri	That's A1
15	Jacob	Yea that's important, she's going to be looking for (at this) . . Alright (Sherri: Not funny)
16	Jacob	The sum of the sides equals 180
17	Sherri	a) No, wait wait area, (N: oh the sum of the) area, b)B is equal to
18	Jacob	Area equals one-half base times height
19	Sherri	a)(under breath: z squared over 4) b)Wait what? The first A
20	Jacob	I know, area equals one-half base times height
21	Sherri	Wait, wait no this is saying, your first A, x=y, cause its isosceles (Jacob: mm hmm) and you know x=y. Oh this one's really easy, Look x=y for A. And you know the Pythagorean theorem. A2 you can set
22	Jacob	Oh we can get rid of a whole , yea
23	Sherri	Yea, so then y squared plus y squared equals z squared, which that equals (more to herself) 2 y squared equals.. .
24	Jacob	a) You could write x squared plus x squared equals z squared. b) Do you object?
25	Sherri	What?
26	Sherri	I don't. . .
27	Jacob	Okay, We're just different 2 x squared equals z squared
28	Sherri	Yea, Wait, so then, y is equal to the square root of 2 z squared, so then we could go back in this equation and put it in, for square root of 2 z squared plus square root of z squared.

29	Jacob	I was a. ..Don't you have to divide by two?
30	Sherri	Oh that's what I meant, yea, (N: Okay) sorry my bad (N: That's okay, it's quite alright)
31	Jacob	I know where you're going
32	Sherri	You get my drift.
33	Jacob	I do. That I do
34	Sherri	equals z squared, so you square these terms and get z squared over two plus z squared over two equals z squared. So you add these together, so you get z squared equals z squared (Laughter. . . .)
35	Sherri	Apparently that wasn't right.
36	Jacob	Right direction
37	Sherri	Right direction, okay.
38	Sherri	So then y equals
39	Jacob	Oh.
40	Sherri	We went in a big circle and got back to it (N: Yea). . .oh lord
41	Jacob	I'm going to have to say. . . I'm going to start working backwards
42	Sherri	Wait, area equals. . . one-half base times height .
43	Jacob	See, Check it out, here's what I'm going to do
44	Sherri	The base and the height are the same: Yea so then (Sherri is talking to herself while Jacob is talking)
45	Jacob	Yea, here's what I'm doing
46	Sherri	have x times
47	Jacob	z squared over 2 squared. . see check that, and we can even go a step further,(A: wait), a= z over two, the squares out there. How'd you like that. It's wild check that out
48	Sherri	a)(one half x squared, should be area, one half x squared---this under her breath to herself)(then louder she says) the area equals one half x squared. . b)wait what are you doing?
49	Jacob	this so far and then I've got
50	Sherri	But z is not. . how do you get this?
51	Jacob	z squared over 4 is this
52	Sherri	is z over two quantity squared
53	Jacob	do you see where I got that now,
54	Sherri	Okay yes (mumbles something inaudible)
55	Jacob	Then I know that one-half x squared equals
56	Sherri	equals the area
57	Jacob	and I also know that this and I can substitute this in for x and we will get that
58	Sherri	a) wait, where's our x, oh x right here b) I'm going to change all of these to x (under breath: x, x).
59	Sherri	a) x equals this. . so then area equals this, so then you plug in , b)wait, I'm confused (more talking to herself)

60	Jacob	You're confused, that's not good.
61	Sherri	a) Oh these equal each other, one half x squared equals z squared over 4, b)(Jacob: You freakin rule we freaking got it) c) so you multiply it and get. . . wait what are you trying to figure out? X squared equals d)Jacob: I can skip this step I don't need this) one half z squared. Square root of x,
62	Jacob	a) and then I just distribute, b)we are awesome, didn't you get it yet
63	Sherri	I'm confused.
64	Jacob	I'll show you, alright.
65	Sherri	Wait let me start over.
66	Jacob	Alright
67	Sherri	which would be one half area x squared, this is equal to area (Jacob: Alright)
68	Sherri	Okay, so this is where we left off.
69	Jacob	Yes that's very important that's crucial
70	Sherri	so x equals that, so you just put in x. . oh that's easy
71	Jacob	Yea, you got it
72	Sherri	equals. . . one-half square root of two z squared, so this is equal to. . .Nuh uh, this doesn't work out
73	Jacob	What, check that out, it works
74	Sherri	I got z squared
75	Jacob	It is supposed to be z squared over 4
76	Sherri	I didn't get 4
77	Jacob	Um,
78	Sherri	x which I square it . . and then it's one half x squared
79	Jacob	Yea remember its supposed to be z squared over two
80	Sherri	Oh, oh that's my problem
81	Sherri	Why did I. .. okay time out we're going to start over (Jacob: Alright), x=y and area equals one-half x, y. A = one half x squared. A1 = x squared plus y squared equals z squared:
82	Jacob	(pirate movies)
83	Sherri	(quiet, talking to self) x squared plus x squared equals z squared so 2 x squared equals z squared So x equals square root of z squared over two. Alright then B, z squared over 4. which that equals but then it's Area equals one half x squared
84	Jacob	what's this, hmm instructions these are fancy recorders. . . are they digital? I wonder if they work like a talkboy
86	Jacob	Paragraph form do we have to do that? Well we should probably justify what we did.
87	Jacob	Umm, definition of isosceles triangle
88	Sherri	oh , substitution, substitution, substitution, substitution simplifying

Appendix F

Transcript for Excerpt B

1	Sara	Okay so we're figuring out
2	Lisa	What's our key question?
3	Sara	Yep what's our key question
4	Sara	How do I show a number is odd?
5	Lisa	Yep
6	Sara	Abstract answer (pause) that would be that n= 2k+1
7	Lisa	Yea, sure
8		Some off task discussion here about their evening plans.
9	Sara	. . . . odd integers Okay so A is that.
10	Lisa	a)Do we need our applied answered ? b)That's the same thing.
11	Sara	a)Yea, that's just n squared is an odd integer. Right Yea? b)Okay so A is n is an odd integer. B is then n squared is odd. Then if you know n is odd then n is equal to 2k + 1. c) I'm writing this in my notebook first because I have no idea what I'm doing.
12	Lisa	Are we working from the bottom or the top first? I don't think it matters so which one?
13	Sara	a)Um I feel like it goes somewhere from the top cause you can't really do anything from the bottom so starting from the top. b)Is that alright with you
14	Lisa	Hmm hmm
15	Sara	Just let me know if. . .
16	Lisa	No, Cause basically everything you've said I've thought the same thing so I've thought well there's no reason to interrupt her. We're thinking on the same wavelength
17	Sara	N squared equals 2 k plus one squared
18	Lisa	Was that 2k plus one?
19	Sara	Yea
20	Lisa	Should we prove n is even first and then if you add a one it's odd? But that would be backtracking that really doesn't make any sense.
21	Sara	yea
22	Lisa	Was it 4 k squared plus 4k plus 1 ?
23	Sara	Yea I had 2k , um I didn't. even do that I always get it wrong. I can't do it in my head. Well I can but I double check it.
24	Lisa	Okay, Well you still got it wrong (lots of laughing from both girls about that being "so mean")
25	Sara	Um. . Are you tired
26	Lisa	I am
27	Sara	You can take a two out of the first term. Showing that one is even. You can take a two out of the second term showing that one is even. You can't take a two out of the last one.
28	Lisa	Right, so that would make it odd.

29	Sara	Yea. . . alright, mathematical marvel (laughing)
30	Lisa	I don't even know, what that is? It sounded fun.
31	Sara	I don't know how you would show that you can take a two out of the middle, two out of this one (???) gonna give you the answer. . . but you can't take a two out of this one and you add that one in and its going to be odd
32	Lisa	a) I understand it but I don't know how to show it, that's the only problem. b)Well like how did we do the first one because ?
33	Sara	
34	Lisa	Well I only wrote down a piece of it.
35	Sara	???Which is what I think we're going to need cause???
36	Lisa	
37	Sara	Thank you
38	Lisa	
39	Sara	That's okay. You're a mathematical marvel.
40		
41	Lisa	Well actually we're here
42	Sara	So we're going to square both sides and that's just math oh just algebra
43	Lisa	Just math
44	Sara	So
45	Lisa	Wait are we . . oh oh
46	Sara	Am I right?
47	Lisa	Yea, I just I. . . we're good
48	Sara	Okay
49	Lisa	But wait, don't we have to say that n is equal to
50	Sara	I did
51	Lisa	Oh, Sorry I'm a little bit behind
52	Sara	No no, you're not a little bit behind at all
53	Lisa	okay
54	Sara	So if we square this and n squared is equal
55	Lisa	4 k squared plus 4k plus one
56	Sara	Hmm hmm
57	Lisa	That's just. . .like and. . .
58	Sara	And this is algebra and this is definition and then. .
59	Lisa	Yea
60	Sara	Okay good, we're four for four. . alright, so we can show that 4k squared is equal to 2 (2k) squared (Lisa choruses in on the 2 (2k) squared) showing that it's even, definition of even number
61	Lisa	Right, I was wondering where you we're going with that.
62	Sara	I was going somewhere
63	Lisa	I was a little confused I didn't hear the second two so I was like wait
64	Sara	So same thing with the next one 4k is equal to two times 2k which is the definition of even . . . Mmm kay, Can you say that one is an

		odd number?
65	Lisa	I don't think you have to prove it, it's not divisible by two. . . that definition.
66	Sara	So one divided by 2 has a remainder
67	Lisa	Right
68	Sara	Oh good, good!
69	Sara	Okay, so one divided by two has a remainder which means that. .
70	Lisa	It's not even
71	Sara	Yea, what do we write down as the reason for that?
72	Lisa	Well its still a definition
73	Sara	Definition of even.
74	Lisa	Yea
75	Sara	Just not using it like we were before
76	Lisa	Right
77	Sara	So. .
78	Lisa	We're proving its not even
79	Sara	Yea. So 4k squared plus 4k would give you an even answer, equals even answer which is using definition of even .
80	Lisa	Wow we're using this definition a lot
81	Sara	a)Yeah, b)So if you're using this definition . . .an even plus one is odd, is an odd answer, do we have to prove that?
82	Lisa	Wait, what if would it be how are we saying, oh gosh
83	Sara	Yea, see we were good to here, now we just
84	Lisa	Let's see if we can start from the bottom and somehow get to that
85	Sara	Okay and finish it off.
86	Lisa	Too bad I just ran into. . .
87	Lisa	Wait why are we using n squared is odd
88	Sara	Because n was an odd integer
89	Lisa	Ohhh
90	Lisa	Why is it n squared again?
91	Sara	? cause that's what we're supposed to do?
92	Lisa	Oh its that oh
93	Sara	An odd number divided by two has a remainder Right?
94	Lisa	Right
95	Sara	And if we have two numbers that don't have a remainder and a number that has a remainder then when you add them up and divide by two you're going to have a remainder right?
96	Lisa	Right
97	Sara	Ding ding
98	Lisa	It is ? but I got it.
99	Sara	Okay, so write it out Lisa
100	Lisa	I tried
101	Sara	No wait. Can't we do this? 4 k squared plus 4k
102	Lisa	
103	Sara	No, no, no No watch You can pull a 2k out of both of those terms.

104	Lisa	Right
105	Sara	Okay now you've shown that these timed together are even , okay, then, now we're still trying to prove that adding an odd number to an even number is an odd
106	Lisa	I feel like…
107	Sara	I feel like we're going in circles. .
108	Lisa	Yea
109	Sara	I feel like we hit it and we don't know how to say it.
110	Lisa	Yea cause like you could divide that by 2 and it's still going to be a remainder, it can't be even. Is there like a definition that says if there's a remainder its going to be odd?
111	Sara	Yeah well its just the opposite of what the even one says
112	Lisa	Then can't we just say that it's odd?
113	Sara	Okay, No I'm
114	Sara	Definition of even
115	Lisa	Definition of even, definition of odd. So therefore
116	Sara	Negating an even definition gives you definition of odd
117	Lisa	a)I know b) but I just don't know how to go from this
118	Sara	I don't think it matters
119	Lisa	Maybe we should of did this from down here You know what I mean, we should of said this is from the bottom up, yea, I think we should of
120	Sara	Alright go, show me what you mean
121	Lisa	No it's the same thing. I'm just saying that we say it's from the bottom up. Just reverse it because then
122	Sara	It reverses it
123	Lisa	Yea, Then because n is odd because we have that
124	Sara	Because we made our definition odd
125	Lisa	Yea
126	Sara	I agree
127	Lisa	It makes more sense that way I think
128	Sara	Okay
129	Lisa	I think
130	Sara	I agree
131	Sara	I say it works
132	Me	How are you guys doing
133	Sara	What we have written, we feel like it makes more sense to go backwards
134	Lisa	Yea even though we have it up here we feel like this should be down here and it should go the opposite way up
135	Me	You said 4k squared is even and 4k is even. . the only other thing I would
136	Sara	That's one divided by two so it has a remainder so its not even making it odd

137	Me	Oh, okay, so you have an even plus an odd, I think your thought process is completely there. So you have an odd, you're saying
138	Sara	But instead of saying like this if we would have gone backwards it would have made more sense, so
139	Me	How are you saying you would work backwards?
140	Sara	Cause we've proved we have an odd number now after showing that this number squared is . . so that would be an odd number for up here. And work like that it would have made more sense. It's hard to explain.
141	Lisa	And then we still want it saying that this was squared somehow, so it probably works out the same.
142	Me	I think the thought process is there, I might suggest that if you

Appendix G

Transcript for Excerpt C

1	Jacob	Use the definition of an isosceles triangle to prove that.. .
2	Jacob	Do you (???)
3	Patrick	Mhmm It's not in writing yet.
4	Jacob	That's good, it took me a while
5	Patrick	It says use the definition of an isosceles triangle to prove that if the right triangle *UVW* with sides of lengths *u* and *v*, and length *w* satisfies sin of *U* equals the square root of *u* over *2v*
6	Jacob	Alright
7	Patrick	I have no idea what I am doing because I haven't taken calc
8	Jacob	Okay, Isosceles triangle's not calc? Isosceles triangle's not calc.
9	Patrick	Yea but the sin of crap like that is
10	Jacob	That's trig
11	Patrick	I haven't taken that either, I took stats.
12	Jacob	Oh okay, Alright. Here's what we're going to do.
13-19	Jacob and Patrick	Off task discussion
20	Jacob	The sin of *U*, Do you want me to give you a quick trig lesson?
21	Patrick	Sure why not, it has to do something with degrees above the horizontal
22	Jacob	We're not worried about that in this instance And our *A* is triangle uvw is isosceles.
23	Patrick	So you're writing it as 2 proofs then?
24	Jacob	Just one
25	Patrick	Cause *A* is that it's a right triangle *and* satisfies this equation and the *B* is that it is isosceles. Trying to figure out that it's isosceles based on its satisfaction of that equation
26	Jacob	Oh really
27	Patrick	and the fact that it's right
28	Jacob	Oh, alright
29	Patrick	So you're writing it as 2 separate proofs. . . .you have like an *A* and an *A* sub B.
30	Jacob	I'm going to make a picture first. . .. Is my picture right so far?
31	Patrick	Is it a right triangle? With 3 sides? W is the hypotenuse
32	Jacob	A triangle has 3 sides? (laughs)
33	Patrick	I don't know I mean you're drawing it.
34	Jacob	Mmm
35-39	Patrick and Jacob	Off task discussion
40	Jacob	Oh, and then sin
41	Patrick	Off task related to 35-39

42	Jacob	Equals opposite over hypotenuse. Alright, what that means like this angle,
43	Patrick	hmmhmm
44	Jacob	we're trying to find the sin of that angle it will be *u* will be opposite
45	Patrick	hmmhmm
46	Jacob	over the hypotenuse *w*
47	Patrick	so
48	Jacob	So then this would be
49	Patrick	So the angle is equal to the length of the opposite side divided by the length of the hypotenuse
50	Jacob	The length of the opposite side divided by the hypotenuse (says it in such a way that he is thinking about it). . .Yeah
51	Patrick	Okay. . .converting angles into lengths and vice versa
52	Jacob	a) Hmm hmm exactly right
52b-60a	Patrick and Jacob	Off task discussion
60	Jacob	b) so in order for this to be true
61	Patrick	It needs to satisfy that equation and be right and then be isosceles
62	Jacob	*W* has to equal 2v Right?
63	Patrick	Right
64	Jacob	It has to satisfy that equation and be right, alright. . . . Oh wait, if this is right and it's isosceles. This has to be 45 degrees, this has to be 45 degrees. . . Where am I going with this? (Long pause) Oh wait, square root, where did the square root come from? Pythagorean theorem maybe? *U* squared plus *v* squared equals *w* squared. And if it's isosceles *a* will, or *b* will equal *u* so it will be two *u* squared or should we go with *v* squared?
65	Patrick	2 *u* squared equals *w* squared.
66	Jacob	I put *v* because over here we have *v*, 2 *v's*.
67-77	Patrick and Jacob	Off task discussion
78	Jacob	Oh wait, and then square root of this, 2v squared, square root of 2v squared equals *w*. And then we have up here 2, oh wait that's like going in a big circle.
79	Patrick	What is?
80	Jacob	Wait! Oh no. I was going to set this *w* equal to this *w*. Right?
81	Patrick	What?
82	Jacob	But. . . it seems unnecessary. .. oh. . . oh wait. . . . hmm
83-90	Patrick and Jacob	Off task discussion
91	Patrick	So, 2 *v* squared plus *w* squared or 2 *v* squared equals *w* squared so (pause)

92	Jacob	What else do we know about isosceles triangles? (long pause)
93	Jacob	Oh I got this tape recorder sitting right in front of me
94	Patrick	It's cool, don't worry about it
95	Jacob	The sin of *u* , okay, yeah that's what I wrote out, wait
96	Jacob	Where'd I get the *w 2v* from? I don't think that's something. ..
97	Patrick	Uhh . . you got it from that . . yeah, plug it in , I don't how
98	Jacob	Oh, I know what I did, it was bad
99	Patrick	I don't know what you did
100	Jacob	Alright, here's what
101	Patrick	So it would be square root of 2 equals w, square root of 2v
102	Jacob	Yeah, like I had down here. This is bad (erasing)
103	Patrick	I have 2 *v* squared equals *w* squared
104	Jacob	If we plug into this it should be *u* wait, sin, I guess we call this angle *u*, angle *u,* okay, equals, *u* over *w*, this is hypotenuse, and then
105	Patrick	Where's the square root come in?
106	Jacob	Oh yea, square root of *2v* squared equals *w* and then we can plug that sin of *u* with *u* over *2v* squared. Square root of *2v* squared. Following me so far?
107	Patrick	A little bit
108	Jacob	I'm bleeding again. That napkin. Oh yea so what was that. Take this and plug it in to the w. Alright, We have to somehow get square root of *u*. . . . Oh and we need to get rid of the squares.
109	Patrick	Yea, I have no idea.
110	Jacob	Well this would it be the same as *u* over 2, square root of 2 v. If we can find an expression for *u* over *v,* just maybe we solve for, there on the same side so that's not going to work. (long pause) I never even finished isosceles. Isos. . . isosceles.
111-131	Patrick and Jacob	Off task discussion
132	Me	How we doing? I'm making my rounds.
133	Patrick	We're a little
134	Jacob	We got here
135	Me	On the first one
136	Jacob	Yea
137	Patrick	It would probably help if I had any expertise in this area, but Jacob's the only one that has taken trig.
138	Me	That's right, and remember I want you to get a trig book and do some trig preparation
139	Patrick	Over Christmas
140	Me	Okay, Do you need a trig book? Do you have one? I might have an extra one in my office.
141	Patrick	I have no idea I'll check around
142	Me	Maybe before Christmas. There might be one in the library that you could check out too. Okay

143	Jacob	So we have sin is opposite over hypotenuse
144	Me	Okay
145	Jacob	And here's side *u* and we're assuming that it's isosceles and a right triangle.
146	Me	Okay, now can you assume that it's isosceles from the start?
147	Patrick	No cause that's B. We can assume it's right. (long pause)
148	Patrick	Who's turn is it to say something?
149	Jacob	Why you all looking at me. I was just rereading the problem
150	Me	Okay, go ahead.
151	Jacob	Um, using the definition of an isosceles triangle to prove that it is right. . . oh so we have to prove that it is right.
152	Patrick	If right and that equation then isosceles
153	Jacob	Oh, okay, if it's right and this equation and by starting with isosceles
154	Me	So what's our key question?
155	Jacob	How do you show a right triangle?
156	Me	Are we trying to show that the triangle is a right triangle?
157	Patrick	Isosceles
158	Jacob	Oh so I guess isosceles?
159	Patrick	How do we show isosceles triangle?
160	Jacob	Alright
161	Me	That doesn't sound like a convinced alright. I'm not convinced that you are alright. Should I be?
162	Patrick	Are you alright?
163	Jacob	I don't really know to tell you the truth?
164	Patrick	I don't know if you're alright either.
165	Me	Okay, it may be, that this part, the instructions are throwing you off a little bit. But the theorem that we're trying to prove, starts here. *If* the right triangle *UVW* with sides of length *u* and *v*.
166	Jacob	Okay, so we know it's a right triangle
167	Me	And hypotenuse.. . Right you know that, you're assuming that, you're also assuming that they hypotenuse of length *w*, so the hypotenuse is side *w*. And they're also telling you that for this triangle *sin u* is equal to that.
168	Jacob	Okay
169	Me	*Then*, so the then part the B part, If A then B, is we want to be able to show that the triangle is
170	Jacob	Isosceles
171	Me	Isosceles, so your key question
172	Jacob	Oh
173	Me	Is How do I show a triangle is
174	Jacob	Isosceles
175	Me	Isosceles Okay, they want you to use the definition to do this proof. So what's the definition of isosceles triangle?
176	Patrick	The definition of isosceles triangle is . .
177	Me	So if you ask the question how do I show the triangle is isosceles

178	Patrick	We could define isosceles
179	Me	Right, which is what
180	Jacob	Uh, 2 sides are
181	Patrick	Equal, or 2 angles
182	Me	So you could show the 2 angles are equal. Okay, so I would jot that down because it's going to guide you. Key question, abstract answer
183	Jacob	Alright
184	Me	And then, I'm kind of pushing you in the direction you have your key question, you have your abstract answer, what's your applied answer? Specifically for this triangle when you're working in the proof, what 2 sides do you want to show are equal?
185	Jacob and Patrick (in unison)	u and v
186	Me	Okay, so this is going to be your conclusion. This is isosceles. So the step right before that is going to be
187	Jacob	u=v
188	Me	So your B1 right here, we're assuming, or that's at least getting us in the right direction. Right. Assuming, we're thinking ahead that *u* is going to be equal to *v*.
189	Jacob	Right
190	Me	Okay, now all this stuff in the "if" part
191	Jacob	Okay
192	Me	Is A, right? If A then B.
193	Jacob	Right
194	Me	So you can use this picture, you can use all this stuff right here, as your premise, as your hypothesis and work with all of that, your knowledge of right triangles, trigonometry and hopefully work down that you see *u* is equal to *v*.
195	Jacob	Oh thank you , that's a lot more concise
196	Me	Okay I'm going to make the rounds again and mosey around. But when you get to the next two, you're proving the exact same theorem, so the A and B are the same, but instead of using the definition you're using Proposition 1 and Proposition 3 which were in your text. So you can look back at them, so the key question will remain the same, but your abstract answer will be different and of course your applied answer. Okay? Give it a shot.
197	Jacob	Okay, so we know that Triangle UVW is right. And that this equation. So that is that like an and? I'll just use and, and
198	Patrick	Well yeah what else would you use?
199	Jacob	There is nothing else to use
200	Patrick	Off task comment
201	Jacob	Alright, so definition of a right triangle.
202	Patrick	Off task comment

203	Jacob	I'm going to write the Pythagorean thm. for this
204	Patrick	It's a good theorem so write it down
205	Jacob	Hmm. If we solve for *u*, we could plug *u* into the top, that might get us farther.
206	Patrick	*U* equals *w* squared minus v squared
207	Jacob	Square root of
208	Patrick	Or *u* squared *u* squared equals *v* squared
209	Jacob	Oh, okay
210	Patrick	
211	Jacob	And then square root of that
212	Patrick	Which we can expand to w+ v and w-v
213	Jacob	Which we can expand to w. . Right right
214	Patrick	Which really doesn't get us anywhere
215	Jacob	Right, so *u* equals square root of *v* squared minus *w* squared. Right? No w minus *v*. Will that get us anywhere?.(Patrick mumbles "w squared minus v squared) . . . Is that a legal operation
216	Patrick	What
217	Jacob	Taking the square root of each term?
218	Patrick	Yeah you gotta take the square root of each side
219	Jacob	Oh, okay
220	Patrick	Square root of the quantity w squared minus v squared
221	Jacob	Okay, yeah. . the difference of two perfect squares

A non-participant enters the discussion here. The conversation continues with the men working on the algebra.

Appendix H

Transcript for Excerpt D

1	Lisa	What the heck am I doing?
2	Patrick	It’s Page 9 in the orange book.
3	Lisa	I don’t have the orange book
4	Patrick	First page in chapter 2 or rather first page in forward backward method. Proposition 1. If XYZ has area of z squared over 4 and it’s a right triangle then it’s isosceles.
5	Lisa	So, Key question
6	Patrick	If I recall we already proved this
7	Lisa	I believe so
8	Patrick	So we can just assume that it’s true for purposes of this proof
9	Lisa	So are we trying to prove that it’s isosceles?
10	Patrick	Yea
11	Lisa	So that would be our key question
12	Patrick	I have no idea. I thought the key question was still how do we show a triangle is isosceles?
13	Lisa	Is that not what I just said?
14	Patrick	Yea
15	Lisa	Making sure we’re on the same page here.
16	Patrick	I’m not. I don’t believe in your trivial pages. Three dimensions
17	Lisa	You are making (fun?)
18	Patrick	Yea I do that.
19	Lisa	(Oh great?) Writing “is an isosceles.” Cause you’re basically writing the same thing. I feel like I’m repeating myself.
20	Patrick	Mmm, I can’t figure out the middle (Not sure if this is what he said)
21	Lisa	I can’t even spell isosceles
22	Patrick	Iso sceles Iso sceles
23	Lisa	Our abstract answer
24	Patrick	Proposition 1 Right triangle XYZ has area z squared over 4 is isosceles. (says this slowly as he is presumably writing) Our applied answer
25	Lisa	So we’re saying it *is* isosceles?
26	Patrick	Yea cause we already proved proposition1 like the second week we were here
27	Lisa	????
28	Patrick	I’ don’t know if we were using folders then or not. But I’m pretty sure we already proved this
29	Lisa	I’m pretty sure you’re right. I think they prove it in here too don’t they
30	Patrick	They might
31	Lisa	a) Somewhere in here. B)Applied answer. What was part A. c)I actually think I have it. That may be helpful. Is this it?
32	Patrick	Page 2.23 Yep

33	Lisa	I don't think we ever finished it. Darn
34	Patrick	Oh we finished it. We even have a really bad condensed proof.
35	Lisa	At least you have a condensed proof. We started we didn't get very far.
36	Patrick	How far is not very
37	Lisa	We did not get very far
38	Patrick	You got x equals 2 over 2.
39	Lisa	I'm celebrating
40	Patrick	You should be
41	Patrick	a) Applied answer
41b-52	Lisa and Patrick	Off task discussion
53	Lisa	Um we're finding what A is
54	Me	Okay
55	Lisa	He might know what A is
56	Me	A is the hypothesis and B is the conclusion. That's how I intended to set that up.
57	Lisa	Oh
58	Me	So A is your "if" part and B is your "then" part. These are arbitrary letters, but that is what your book is using.
59	Lisa	Oh, okay.
60	Me	Okay, so what is it that you know
61	Lisa	We know that triangle XYZ is isosceles and has an area of z squared over 4. I think?
62	Me	That was the proposition right, that you're going to use?
63	Lisa	Oh I thought we already proved it so we could assume that it's true
64	Me	Yea, you can assume it's true, yes
65	Patrick	Cause we proved it right there, we proved it the second week.
66	Me	Right. Okay, so you can use that to help you prove this other theorem
67	Patrick	So that yea
68	Me	May I guide you here. How do I show a triangle is isosceles is your key question. Okay, now they say to use the proposition which you've identified, you wrote that down right there. That's good that's going to be your abstract answer. Now the only thing I would push you towards a little bit, because of how you answered my questions just a few minutes ago makes me think its already mumbled and jumbled. Remember abstract answer needs to be very broad and not specific with letters for a specific problem. Okay? So in this triangle that we proved, what was z
69	Lisa	The hypotenuse
70	Me	Okay, so what I would challenge you to do is to write out your answer a little more general or broad and not say z squared over 4 because our triangle, our new triangle doesn't have z. So in words what could you say? Show that the
71	Lisa	Hypotenuse squared divided by 4
72	Me	Divided by 4 Okay and I'm going to keep pushing you a little bit.

		Okay but that's right here
73	Lisa	Oh
74	Me	For your abstract answer it's general
75	Lisa	Oh
76	Patrick	Instead of z it's hypotenuse
77	Lisa	I got it.
78	Me	Okay, Okay, Now I'm going to keep pushing you. You're applied answer is the same exact thing as the abstract answer, only you're going to apply it specifically to the one you're working with. So the triangle that you're working with, how are you going to show it's isosceles
79	Lisa	That the length w squared over 4 is the area.
80	Me	Exactly. So what you want to show is that the area of this triangle is w squared over 4 for the applied answer
81	Lisa	Oh, I keep trying to get to A, I just want to get that one over with. Did you already have this? And you didn't help me We're partners!
82	Patrick	Well I didn't get it down till she already came over and I thought it would be rude to interrupt
83	Lisa	Okay
84	Me	I'm going to move on to the next group. So ultimately you want to show that the area is equal to this and when you do you will have proven that the triangle is isosceles. Now before you move on do you have you're A written down? Good you do so you guys can talk about that. You need to think if you need to show that area is equal to that, what would the next question in your mind be?
85	Lisa	That the area is this equal to that
86	Me	Right and we're trying to find the area of a what
87	Patrick	Right triangle
88	Me	And how do you find the area of a triangle
89	Patrick	One half base times height
90	Me	Okay so that's how you're wheels are spinning so you're going to have your top your bottom and that's the kind of the stuff that's in your mind trying to get them to meet when you're working down and working up. I'll be back
91	Patrick	A is everything after the if up until the then
92	Lisa	No don't write on it
93	Patrick	Okay, that's the A, sorry
94	Lisa	The Right triangle UVW , oh I'm trying to put this
95	Patrick	And also this, because it's important
96	Lisa	The right triangle UVW (sighs) . . .equal to this. How's that look. Is that acceptable? Are you sure
97	Patrick	Well , no cause triangles aren't really equal to anything but it's close enough
98	Lisa	Fine. Then what would you put
99	Patrick	I have "has" cause it's a quality of the triangle

100	Lisa	The right triangle UVW *has* sin. That doesn't make sense
101	Patrick	It would make even less sense to equal it. Cause would you really put this in an equation
102	Lisa	Yes (laughs)
103	Patrick	You'd put that on a triangle. How would you manipulate that mathematically
104	Lisa	I didn't finish the sin u equals, so I can't, okay
105	Patrick	A triangle has no mathematical value
106	Lisa	Here we'll just put what it has up there.
107	Patrick	Are you satisfied? That works.
108	Lisa	My way
109	Patrick	You mean it makes more sense
110	Lisa	No you need to pick. You need to talk good English. Okay so B is
111	Patrick	You mean well English
112	Lisa	We're never going to get this done
113	Patrick	Not the way you're working it
114	Lisa	Me? If you would discuss with me this thing would be going a whole lot faster. She's going to have this whole thing recorded of us arguing the whole time
115	Patrick	Who's arguing I'm having a great time. You're the one arguing. I'd like that last statement stricken from the record. Anyway. Then
116	Lisa	Okay
117	Patrick	UVW is isosceles.
118	Lisa	(is isosceles – simultaneous with Patrick) Woo, glad I can write, okay, what did you have as… okay A
119	Patrick	Hmm
120	Lisa	Why are you going hmm
121	Patrick	I can't go Hmm
122	Lisa	Oh no, I wasn't asking why you were I thought you had an idea.
123	Patrick	No I just go hmm
124	Lisa	Okay, so I guess we should start with what we're given or what we know. So should we start with the sin u equals the square root of u over 2v
125	Patrick	Sure. . . um go for it.
126	Lisa	Wait, . . . okay so, what did we say? Right triangle is one-half base times height. That's not a right triangle. Here we go. Well that the , mmm, I don't, I don't know. Should we start with that? Cause could we start with area and put the w squared over 4? And then cause that's the area of isosceles, but then we could do something with the area of a right triangle. (pause)
127	Patrick	So we have the area equals one half base times height equals w squared
128	Lisa	Over 4
129-163	Patrick and Lisa	Off task discussion

164	Lisa	Okay , so are we making that our second one
165	Patrick	I guess, I already have it down
166	Lisa	I feel like we're not getting anywhere
167	Patrick	Well we'll get there, don't make me turn this proof around
168	Lisa	I don't know what that means
169	Patrick	???
170	Lisa	Kay
171	Patrick	The base is one of the shorter sides and so is the height one half u times v
172	Lisa	Can we say that?
173	Patrick	By this diagram
174	Lisa	Well I understand what you mean but like
175	Patrick	Base and height, just plug them in
176	Lisa	I believe you
177	Patrick	We can ask here when she comes around next but I'm pretty sure it's by Diagram 1 This is now diagram 1 in case anyone asks, see it's got a 1
178	Lisa	I'm very informed
179	Patrick	That's good to hear. Multiply both sides by 4 to get
180	Lisa	2
181	Patrick	2uv equals w squared
182	Lisa	Do we square them all? Or take the square root of them all? No that wouldn't be any fun
183	Patrick	No I think we should take a page out of her book and disassociate the w altogether and put in u squared plus v squared
184	Lisa	Okay
185	Patrick	I don't know if that will work, I'm hoping it will work
186	Lisa	Hmm, well, let's think. Can we somehow get, no I don't think we can . Now wait sin u is the square root of u over 2v. What does that also equal. U over
187	Patrick	W
188	Lisa	u over w?
189	Patrick	Mm hmm
190	Lisa	Can we get this equal u over w?
191	Patrick	No cause they're measuring two different quantities
192	Lisa	Darn
193	Lisa	That's measuring the area and that's measuring the angle
194	Patrick	Darn. Well I thought I'd give it a try
195	Patrick	At least I think it's measuring the angle
196	Lisa	Oh no
197	Patrick	Something about the angle
198	Lisa	Oh no
199-200	Patrick and Lisa	Off task
201	Me	How's it going

202	Lisa	He's talking about (off task topic) I'm not really sure what we're going this
203	Patrick	I'm not quite sure either
204	Me	Your applied answer was show that area is equal to w squared over 4. Right. And did I challenge you to thing about the area of a triangle? Area is one half base times height. Okay, now I see you having A's written down which is fine, but to me the stuff you have written down are answer to the key question, right, which is the backward process
205	Lisa	Oh
206	Me	Okay, so It's not wrong, what we label these A's is arbitrary but what I would do is I'd start with this A is sin u right here
207	Lisa	Mmm hmm
208	Me	and then my B down here is what was your applied answer, show area equals w equals w squared over 4. So needs to be area of this triangle is w squared over 4. B needs to be your last step
209	Lisa	So this triangle
210	Me	Right the area of triangle UVW . . . Okay, and how do we show that? The step before it you'll show one half uv equals . . . okay, so now what I would do is
211	Lisa	Yea
212	Me	Erase this at least and the idea is can you work with this and get down and get that and when you do you'll have it
213	Lisa	Okay
214	Me	Okay
215	Lisa	Well that makes sense.
216	Patrick	I don't do anything in pencil.
217	Lisa	No big deal, let's try again. What did she do, I'm sorry we're you looking at this?
218	Patrick	No What did she do
219	Lisa	Do we want to set these two equal to each another and see if that will somehow get us here
220	Patrick	I'm not sure how they could, like I said they're measuring different quantities.
221	Lisa	Well you figure they have to somehow because if we're starting with this we're somehow going to have to use what's given to get this.
222	Patrick	Mmm hmm
223	Lisa	Because this one right here, that's the angle, but these are the lengths. These are the lengths too, so if we can get the lengths equal here equal here then it will work. Know what I mean? Or am I confused
224	Patrick	I'm perpetually confused
225	Lisa	Well I guess that makes sense a little better, I'm not sure though
226	Patrick	I'm not sure how that could be (??)
227-230	Lisa and Patrick	Off task
231	Lisa	Now wait hold on. I feel like this needs to be in these steps. Okay then

		we can say that this equals to this. Now wait!
232	Patrick	???
233	Lisa	I don't know hold on. Let me see???
234	Patrick	
235	Lisa	Oh yea????
236	Patrick	. . .so I can read it
237	Lisa	I'm trying
238	Me	Okay, do you really want to work with that square root
239	Lisa	No
240	Me	No so what would you do to both sides
241	Lisa	Square them
242	Patrick	Guess an idea?
243	Me	Do you really want to work with 2 fractions like that? Naa
244	Lisa	No
245	Me	So what would you do
246	Lisa	Get rid of them (laugh)
247	Me	Get rid of the fractions ? Yea
248	Lisa	So (???)
249	Me	Is there any way you can simplify that?
250	Lisa	Oh, Probably
251	Me	And actually , this is why forward backwards is so cool and no body else did this step but it's going to help you 2uv equals w squared . What can you do, you're like one link away
252	Lisa	Get rid of Wait,
253	Patrick	Algebra
254	Lisa	Oh we just go like this and those cancel out
255	Me	Yea you did it!
256	Lisa	I did it!!! (Giggle)
257	Me	Now your last task , now that you guys did it is to
258	Patrick	Condense it!
259	Me	Very mathematical like you're a budding young mathematician
260	Patrick	I think you're going to have to help me
261	Lisa	Of course
262	Patrick	I mean you solved it
263	Lisa	(laughing)
264	Patrick	Algebra!
265	Lisa	Oh man I had 2 twice. Are you going to write this down? Or am I just (still laughing)
266	Patrick	I have something similar
267	Lisa	Okay that's good (laughing) No (laughing)
268	Patrick	I have something similar. I got to here from here.
269	Lisa	I'm excited (whispers something)
270	Patrick	Off task comment
271	Lisa	It was good. We get to do our condensed proof now. Oh I will try. Yea all you did was sit there. Yea? (talking across to another group) Oh,

		okay. Oh, Oh yea, I got this.
272	Patrick	What
273	Me	(I talk to whole group and ask them to finish their condensed proof)
274	Lisa	To reach the conclusion triangle UVW is isosceles Oh I spelled isosceles again. Is isosceles, it will . . . equal to. Did you put Oh what are you writing?
275	Patrick	Yea it is, Does it bother you that I'm writing something different? Why does it bother you that I'm writing something different?
276	Lisa	Cause were partners and we're supposed to
277	Patrick	Partners in what? Crime?
278	Lisa	No this is a math problem
279	Patrick	I call that a crime. Smart side
280	Lisa	Given that the right triangle UVW satisfies sin of u equals square root u over 2v equals u over w, we can by algebra, algebra, that, that, Is it okay that I started my like that?
281	Patrick	Yea, a proof is a proof
282	Lisa	Are you sure
283	Patrick	Yea, I'm pretty sure
284	Lisa	Off task comment
285	Lisa	Oh man
286	Patrick	I'm on a roll, do you really want to kill that? (Long pause)
287	Lisa	The hypothesis, right triangle. (off task comment) I cannot, No, I'm having trouble, I can't
288	Patrick	(?J-bra?)
289	Lisa	We didn't use the Pythagorean Theorem in this proof did we? I don't think so.
290	Patrick	Uhh, we could but We didn't have to
291	Lisa	We didn't
292	Patrick	
293	Lisa	Oh
294	Patrick	I didn't mention it
295	Lisa	Laughing
296	Patrick	What
297	Lisa	Oh man. We showed that don't break it

Appendix I

Transcript for Excerpt E

1	Patrick	Page 33. If n is an odd integer, then n squared is an odd integer
2	Patrick	Alright then.
3	Amanda	Okay, Now we have the definition of an odd integer
4	Karen	Definition number 6
5	Amanda	Yea,
6	Patrick	It says its odd if and only if number divided by two has a remainder of 1
7	Amanda	Yea n= 2k +1, so we can say in A1 that n=2k+1 by the definition.. . and then can't we. . .we're allowed to use a proposition, right
8	Karen	I think so , we used it before
9	Amanda	Then we can say n squared is equal to 2k + 1 squared. Now after you do this you'll get 2k squared plus 2k plus 1. Then by proposition 2 this is even and then this would be even too, this would be like 2n
10	Patrick	And an even number plus one is an odd number
11	Amanda	a) Yea exactly, so its going to be odd. But I don't know like what to put over here. Like this would be. . .
12	Patrick	I keep drawing twos instead of ns, I'm sure that is a sign of losing my mind.
13	Amanda	This would be Algebra and then by proposition 2 this would be proposition 2 and that would make this even. Like I just don't know how to like order it and stuff, and what to put over here. And 2k squared is even is proposition two. 2m 2k is even by definition 5. So n squared is equal to an even number. So even plus even.
14		Long pause (about 1:15) (Amanda may be mumbling a few things to herself, but inaudible
15	Patrick	How about this. We know that n squared is equal to this expanded business
16	Both girls	Hmm and Yeah
17	Patrick	We say that the term n squared is equal to this short hand the different 2l times 1. We can set those two equal cancel out the ones. And since we know that two even numbers sum to an even number we're done. Since we can just add the one back in and make them both odd.
18	Girl	Yea
19		
20	(I enter in)	(Patrick says something or asks me what I think or if its okay – exact wording is inaudible
21	Me	I came in half way. . . but it matters what they think. These are your mathematician colleagues right here.
22	Karen	Well it makes sense to me because that's the proposition and we just squared it to get that. And then he just subtracted the one and they'd both be even. And then if you added the one back in they'd both be odd.

23	Amanda	Yea
24	Karen	Cause an even number plus one is odd
25	Me	Okay, is that how you answered, what was your key question?
26	Karen	Hmm, Hmm. We didn't write it down.
27	Me	Oh you didn't write it down.
28	Patrick	We just kind of jumped right in.
29	Me	Well that's okay, go back and think about it now. What was it that you needed to show and have you shown it.
30	Patrick	How do you show a number is odd
	Me	Okay, and how would you have to do that.
31	Karen	Show that it n= 2k+1
32	Patrick	Show that there is a remainder of 1 when you divided by 2.
33	Me	Okay
34	Girl	Wait a minute
35	Me	So that's an even number two times an integer plus one.
36	Patrick	Hmm Hmm
37	Me	So what's your integer then?
38	Karen	n
39	Me	I'm sorry, not your integer
40	Patrick	l or k
41	Karen	Well, yea
42	Me	So you redefined.
43	Patrick	N is 2k +1 and n squared is 2l + 1 and you squared both of these. 2 K + 1 squared is this, which is also this. If we set those equivalent you can show.
44	Me	So this is kind of your backward step then?
45	Patrick	Yea
46	Me	So what I'm asking is if you worked forward to here, can you tell me what your l is?
47	Girls	chuckle
48	Me	This is n squared right?
49	Amanda	Yea
50	Amanda	Your l would be your 2k problem your factoring. Like you ___ this, and this basically saying that he took a two out of here, and then it would be 2k squared. Then instead of putting that in there, he just made another variable for it. Instead of making it more complicated.
51	Patrick	Just condensing things.
52	Karen	I understand
53	Amanda	I get what he's doing
54	Patrick	Well I don't can you explain it to me?
55	Me	And, can you write it in a logical form so that somebody that was seeing if you really did understand would know. Can you convince someone else in a mathematical way?
56	Patrick	This is why I have beautiful translators like these. (joking about recorders)

57	Me	This is why we're having this seminar to help with that. I think you're almost there. May I use this. This is your n squared. I want to push you a bit further to make it a little bit more clear to an audience trying to read this. If this is your n squared and ultimately down here you wanted to get n squared equals 2l + 1 where l is some integer. And I think Amanda was almost there. And you have that equals 2l + 1. I think you want that underneath it. How do I show, What is my l? The plus ones match up. And Amanda said you can factor a 2 out of this quantity right here these two terms.
58	Students	Yea, hmm-hmm, right
59	Me	Okay, so that would leave me with what in parentheses.
60	Patrick	k squared plus k
61	Girls in unison	2k
62	Patrick	Yea, 2 k squared plus 2k
63	Me	And that's equal to
64	Patrick	l
65	Me	Which is these numbers which is l
66	Patrick	odd
67	Me	Well we don't know if l is even or odd, but 2l is even
68	Students	yea
69	Me	Okay, so that's what I want to show. I want to see 2 times 2ksquared + 2k quantity plus one and then you'd make that 2l. But over here you need to explain l. The ideas are there, it's getting it translated on paper.
70	Amanda	If you put 2l +1
71	Me	Then in your justification you'll explain l is an integer because products and sums of integers are integers and that's what the 2k squared plus 2 k was.
72		Extended period of no talking and then some off task discussion
73	Me	I challenge them to write a condensed paragraph proof
74		
75	Karen	Odd integer
76		State the conclusion
77		Long pause
78	Patrick	From the hypothesis that the integer n is odd, a state that can be defined as being one more than an even integer
79	Karen	Comma
80	Patrick	Well naturally
81	Karen	I'm just going to copy you
82	Patrick	From the hypothesis we argue that, that the quantity n squared is also odd.
83	Karen	Comma, we argue
84-		Off task comments